Matrix methods in stability theory

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joint work with Mikhail Tyaglov

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Outline

1. Complex rational functions
   - Hankel and Hurwitz matrices
   - Resultants and their applications
   - Euclidean algorithm

2. Real rational functions
   - Sturm algorithm
   - Cauchy indices
   - Stable polynomials
   - Hyperbolic polynomials
Let $R(z)$ be a rational function expanded in its Laurent series at $\infty$

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots.$$ 

Introduce the infinite Hankel matrix $S := [s_{i+j}]_{i,j=0}^{\infty}$ and consider the leading principal minors of $S$:

$$D_j(S) := \det \begin{bmatrix}
  s_0 & s_1 & s_2 & \cdots & s_{j-1} \\
  s_1 & s_2 & s_3 & \cdots & s_j \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  s_{j-1} & s_j & s_{j+1} & \cdots & s_{2j-2}
\end{bmatrix}, \quad j = 1, 2, 3, \ldots.$$ 

These are Hankel minors or Hankel determinants.
Let $R(z)$ be a rational function expanded in its Laurent series at $\infty$

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These are Hankel minors or Hankel determinants.
Complex rational functions
Real rational functions
Hankel and Hurwitz matrices
Resultants and their applications
Euclidean algorithm

Hurwitz determinants

Let \( R(z) = \frac{q(z)}{p(z)} \), \( p(z) = a_0 z^n + \cdots + a_n \), \( a_0 \neq 0 \),
\( q(z) = b_0 z^n + \cdots + b_n \),

For each \( j = 1, 2, \ldots \), denote

\[
\nabla_{2j}(p, q) := \det \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_{j-1} & a_j & \cdots & a_{2j-1} \\
b_0 & b_1 & b_2 & \cdots & b_{j-1} & b_j & \cdots & b_{2j-1} \\
0 & a_0 & a_1 & \cdots & a_{j-2} & a_{j-1} & \cdots & a_{2j-2} \\
0 & b_0 & b_1 & \cdots & b_{j-2} & b_{j-1} & \cdots & b_{2j-2} \\
0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_j \\
0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_j
\end{bmatrix}.
\]

These are the \textbf{Hurwitz minors} or \textbf{Hurwitz determinants}. 

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Matrix methods in stability theory
Theorem [Hurwitz].

Let \( R(z) = q(z)/p(z) \) with notation as above. Then

\[
\nabla_2 j(p, q) = a_0^{2j} D_j(R), \quad j = 1, 2, \ldots.
\]

Corollary.

Let \( T(z) = -1/R(z) \) with notation as above. Then

\[
D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \ldots.
\]
Hankel ↔ Hurwitz

**Theorem [Hurwitz].**

Let \( R(z) = q(z)/p(z) \) with notation as above. Then

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\nabla_{2j}(p, q) = a_0^{2j} D_j(R), \quad j = 1, 2, \ldots.
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**Corollary.**

Let \( T(z) = -1/R(z) \) with notation as above. Then

\[
D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \ldots.
\]
Let $p$ and $q$ be as above and let $b_0 \neq 0$. The resultant of $p$ and $q$ is defined as

$$R(p, q) := \det \begin{bmatrix}
a_0 & a_1 & \ldots & a_{n-1} & a_n & \ldots & a_{2n-1} \\
0 & a_0 & \ldots & a_{n-2} & a_{n-1} & \ldots & a_{2n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_0 & a_1 & \ldots & a_n \\
b_0 & b_1 & \ldots & b_{n-1} & a_n & \ldots & b_{2n-1} \\
0 & b_0 & \ldots & b_{n-2} & b_{n-1} & \ldots & b_{2n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & b_0 & b_1 & \ldots & b_n
\end{bmatrix}. $$
Theorem.  

Given polynomials $p$ and $q$, let $\lambda_i \ (i = 1, \ldots, n)$ be the zeros of $p$, and let $\mu_j \ (j = 1, \ldots, n)$ be the zeros of $q \ (b_0 \neq 0)$. Then

$$
\mathbf{R}(p, q) = \left( -1 \right)^{\frac{n(n-1)}{2}} \nabla_{2n}(p, q) = a_0^n \prod_{i=1}^{n} q(\lambda_i)
$$

$$
= a_0^n b_0^n \prod_{i,j=1}^{n} (\lambda_i - \mu_j) = (-1)^n b_0^n \prod_{j=1}^{n} p(\mu_j).
$$
Definition.
Let a polynomial $p$ have roots $\lambda_i$ ($i = 1, \ldots, n$). The **discriminant** of $p$ is defined by

$$D(p) = a_0^{2n-2} \prod_{j<i}^{n} (\lambda_i - \lambda_j)^2.$$ 

Theorem.
For a polynomial $p$ of degree $n$,

$$R(p, p') = (-1)^{\frac{n(n-1)}{2}} a_0 D(p).$$
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Orlando’s formula

**Theorem [generalized Orlando].**

The resultant polynomials $p$ and $q$ can be computed by the formula

$$R(p, q) = (-1)^{\frac{n(n+1)}{2}} c \prod_{i<k}(z_i + z_k),$$

where $z_i$ are the zeros of the polynomial $h(z) := p(z^2) + zq(z^2)$, and

$$c := \begin{cases} a_0^{m+n} & \text{if } \deg q = m \leq n - 1, \\ b_0^{2n} & \text{if } \deg q = n. \end{cases}$$
Starting from \( f_0 := p, \ f_1 := q - (b_0/a_0)p \), form the Euclidean algorithm sequence

\[
f_{j-1} = q_j f_j + f_{j+1}, \quad j = 1, \ldots, k, \quad f_{k+1} = 0.
\]

Then \( f_k \) is the greatest common divisor of \( p \) and \( q \). This gives a continued fraction representation

\[
R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{q_1(z) + \frac{1}{q_2(z) + \frac{1}{q_3(z) + \cdots + \frac{1}{q_k(z)}}}}.
\]
Euclidean algorithm and continued fractions

Starting from $f_0 := p$, $f_1 := q - (b_0/a_0)p$, form the Euclidean algorithm sequence

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Generalized Jacobi matrices

\[ J(z) := \begin{bmatrix}
  q_k(z) & -1 & 0 & \ldots & 0 & 0 \\
  1 & q_{k-1}(z) & -1 & \ldots & 0 & 0 \\
  0 & 1 & q_{k-2}(z) & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & q_2(z) & -1 \\
  0 & 0 & 0 & \ldots & 1 & q_1(z)
\end{bmatrix} \]

Remark 1. \( h_j(z) \) is the leading principal minor of \( J(z) \) of order \( k - j \). In particular, \( h_0(z) = \det J(z) \).

Remark 2. Eigenvalues of the generalized eigenvalue problem

\[ J(z)u = 0 \]

are closely related to the properties of \( R(z) \).
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\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & q_2(z) & -1 \\
0 & 0 & 0 & \ldots & 1 & q_1(z)
\end{bmatrix}.$$  

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Remark 2. Eigenvalues of the generalized eigenvalue problem

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are closely related to the properties of $R(z)$. 

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Theorem.

If \( R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{q_1(z) + \frac{1}{q_2(z) + \frac{1}{q_3(z) + \cdots + \frac{1}{q_k(z)}}}} \)

with \( n_j := \deg q_j \), we have, for all \( j = 1, 2, \ldots, k \),

\[
D_{n_1+n_2+\ldots+n_j}(R) = \prod_{i=1}^{j} (-1)^{\frac{n_i(n_i-1)}{2}} \cdot (-1)^{\sum_{i=0}^{j-1} in_i+1} \cdot \prod_{i=1}^{j} \frac{1}{n_i+2 \sum_{\rho=i+1}^{j} n_\rho}.
\]
In the regular case,

\[ q_j(z) = \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{C}, \alpha_j \neq 0. \]

The polynomials \( f_j \) satisfy the three-term recurrence relation

\[ f_{j-1}(z) = (\alpha_j z + \beta_j)f_j(z) + f_{j+1}(z), \quad j = 1, \ldots, r. \]

\[ R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{\alpha_1 z + \beta_1 + \frac{1}{\alpha_2 z + \beta_2 + \frac{1}{\alpha_3 z + \beta_3 + \frac{1}{\cdots + \frac{1}{\alpha_r z + \beta_r}}}}. \]
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In the **doubly regular case**, 

\[
q_{2j}(z) = c_{2j}, \quad j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor,
\]

\[
q_{2j-1}(z) = c_{2j-1}z, \quad j = 1, \ldots, r.
\]

\[
R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{c_1z + \frac{1}{c_2 + \frac{1}{c_3z + \cdots + \frac{1}{T}}}}, \quad \text{where}
\]

\[
T := \begin{cases} 
    c_2r & \text{if } |R(0)| < \infty, \\
    c_{2r-1}z & \text{if } R(0) = \infty.
\end{cases}
\]
Stieltjes continued fractions

In the doubly regular case,

\[ q_{2j}(z) = c_{2j}, \quad j = 1, \ldots \left\lfloor \frac{k}{2} \right\rfloor, \]
\[ q_{2j-1}(z) = c_{2j-1}z, \quad j = 1, \ldots, r. \]

\[ R(z) = \frac{f_1(z)}{f_0(z)} = \frac{1}{c_1 z + \frac{1}{c_2 + \frac{1}{c_3 z + \cdots + \frac{1}{T}}}}, \]

where

\[ T := \begin{cases} 
    c_{2r} & \text{if } |R(0)| < \infty, \\
    c_{2r-1}z & \text{if } R(0) = \infty.
\end{cases} \]
The associated generalized eigenvalue problem:

$$(A z + B) u = 0,$$

where

$$A = \begin{bmatrix}
\alpha_r & 0 & \ldots & 0 & 0 \\
0 & \alpha_{r-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \alpha_2 & 0 \\
0 & 0 & \ldots & 0 & \alpha_1
\end{bmatrix}, \quad B = \begin{bmatrix}
\beta_r & -1 & \ldots & 0 & 0 \\
1 & \beta_{r-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \beta_2 & -1 \\
0 & 0 & \ldots & 1 & \beta_1
\end{bmatrix}.$$
Connections to infinite Hurwitz matrices

\[ H(p, q) := \begin{bmatrix}
  a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\
  b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\
  0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
  0 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  0 & 0 & a_0 & a_1 & a_2 & a_3 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \quad (\text{deg } q < \text{deg } p). \]

\[ H(p, q) = \begin{bmatrix}
  a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\
  b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\
  0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
  0 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  0 & 0 & a_0 & a_1 & a_2 & a_3 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \quad (\text{deg } q = \text{deg } p). \]
Theorem.

If \( g(z) = g_0 z^l + g_1 z^{l-1} + \ldots + g_l \), then

\[
H(p \cdot g, q \cdot g) = H(p, q) T(g),
\]

where

\[
T(g) := \begin{bmatrix}
  g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \\
  0 & g_0 & g_1 & g_2 & g_3 & \cdots \\
  0 & 0 & g_0 & g_1 & g_2 & \cdots \\
  0 & 0 & 0 & g_0 & g_1 & \cdots \\
  0 & 0 & 0 & 0 & g_0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Here we set \( g_i = 0 \) for all \( i > l \).
Another factorization

Theorem.
If the Euclidean algorithm for the pair $p, q$ is doubly regular, then $H(p, q)$ factors as

$$H(p, q) = J(c_1) \cdots J(c_k) H(0, 1) I(g),$$

where

$$J(c) := \begin{bmatrix} c & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & c & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & c & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad H(0, 1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
Sturm’s algorithm is a variation of the Euclidean algorithm

\[ f_{j-1}(z) = q_j(z)f_j(z) - f_{j+1}(z), \quad j = 0, 1, \ldots, k, \]

where \( f_{k+1}(z) = 0 \). The polynomial \( f_k \) is the greatest common divisor of \( p \) and \( q \).

The Sturm algorithm is regular if the polynomials \( q_j \) are linear.

The Sturm algorithm was originally proposed to count zeros on a real interval.
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Cauchy indices

Definition.

\[ \text{Ind}_\omega(F) := \begin{cases} 
+1, & \text{if } F(\omega - 0) < 0 < F(\omega + 0), \\
-1, & \text{if } F(\omega - 0) > 0 > F(\omega + 0), 
\end{cases} \]

is the index of the function \( F \) at its real pole \( \omega \) of odd order.

Theorem [Gantmakher].

If a rational function \( R \) with exactly \( r \) poles is represented by a series

\[ R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots, \]

then

\[ \text{Ind}_{-\infty}^{\infty} = r - 2S(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)). \]
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\]

then

\[
\text{Ind}_{-\infty}^{+\infty} = r - 2S(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)).
\]
Definition.

A polynomial is **stable** if all its zeros lie in the left half-plane.

Theorem.

A polynomial $f = p(z^2) + zq(z^2)$ is stable if and only if its infinite Hurwitz matrix $H(p, q)$ is totally nonnegative.
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A polynomial is **hyperbolic** if all its zeros are real.

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