# Matrix methods in stability theory

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# Outline

1. Complex rational functions
   - Hankel and Hurwitz matrices
   - Resultants and their applications
   - Euclidean algorithm
2. Real rational functions
   - Sturm algorithm
   - Cauchy indices
   - Stable polynomials
   - Hyperbolic polynomials

# Hankel and Hurwitz matrices

Let $R(z)$ be a rational function expanded in its Laurent series at $\infty$

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots .$$

Introduce the infinite Hankel matrix $S := [s_{i+j}]_{i,j=0}^{\infty}$ and consider the leading principal minors of $S$:

$$D_j(S) := \det \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-2} \end{bmatrix}, \quad j = 1, 2, 3, \dots .$$

These are Hankel minors or Hankel determinants.

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# Hurwitz determinants

$$\text{Let } R(z) = \frac{q(z)}{p(z)}, \quad p(z) = a_0 z^n + \cdots + a_n, \quad a_0 \neq 0,$$
$$q(z) = b_0 z^n + \cdots + b_n,$$

For each $j = 1, 2, \ldots$, denote

$$\nabla_{2j}(p,q) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{j-1} & a_j & \ldots & a_{2j-1} \\ b_0 & b_1 & b_2 & \ldots & b_{j-1} & b_j & \ldots & b_{2j-1} \\ 0 & a_0 & a_1 & \ldots & a_{j-2} & a_{j-1} & \ldots & a_{2j-2} \\ 0 & b_0 & b_1 & \ldots & b_{j-2} & b_{j-1} & \ldots & b_{2j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_0 & a_1 & \ldots & a_j \\ 0 & 0 & 0 & \ldots & b_0 & b_1 & \ldots & b_j \end{bmatrix}.$$

These are the Hurwitz minors or Hurwitz determinants.

# Hankel $\leftrightarrow$ Hurwitz

**Theorem [Hurwitz].**

Let $R(z) = q(z)/p(z)$ with notation as above. Then

$$\nabla_{2j}(p,q) = a_0^{2j} D_j(R), \quad j = 1, 2, \dots.$$

**Corollary.**

Let $T(z) = -1/R(z)$ with notation as above. Then

$$D_j(S) = s_{-1}^{2j} D_j(T), \quad j = 1, 2, \dots.$$

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# Resultant

Let $p$ and $q$ be as above and let $b_0 \neq 0$. The resultant of $p$ and $q$ is defined as

$$\mathbf{R}(p,q) := \det \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n & \dots & a_{2n-1} \\ 0 & a_0 & \dots & a_{n-2} & a_{n-1} & \dots & a_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_{n-1} & a_n & \dots & b_{2n-1} \\ 0 & b_0 & \dots & b_{n-2} & b_{n-1} & \dots & b_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & b_n \end{bmatrix}.$$

# Resultant formulæ

**Theorem.**

Given polynomials $p$ and $q$, let $\lambda_i$ $(i = 1, \ldots, n)$ be the zeros of $p$, and let $\mu_j$ $(j = 1, \ldots, n)$ be the zeros of $q$ $(b_0 \neq 0)$. Then

$$\begin{aligned}\mathbf{R}(p,q) &= (-1)^{\frac{n(n-1)}{2}} \nabla_{2n}(p,q) = a_0^n \prod_{i=1}^{n} q(\lambda_i) \\ &= a_0^n b_0^n \prod_{i,j=1}^{n} (\lambda_i - \mu_j) = (-1)^n b_0^n \prod_{j=1}^{n} p(\mu_j).\end{aligned}$$

# Resultant → discriminant

**Definition.**

Let a polynomial $p$ have roots $\lambda_i$ $(i = 1, \ldots, n)$. The discriminant of $p$ is defined by

$$\mathbf{D}(p) = a_0^{2n-2} \prod_{j<i}^{n} (\lambda_i - \lambda_j)^2.$$

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For a polynomial $p$ of degree $n$,

$$\mathbf{R}(p, p') = (-1)^{\frac{n(n-1)}{2}} a_0 \mathbf{D}(p).$$

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# Orlando’s formula

**Theorem [generalized Orlando].**

The resultant polynomials $p$ and $q$ can be computed by the formula

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n+1)}{2}} c \prod_{i<k} (z_i + z_k),$$

where $z_i$ are the zeros of the polynomial $h(z) := p(z^2) + zq(z^2)$, and

$$c := \begin{cases} a_0^{m+n} & \text{if } \deg q = m \le n-1, \\ b_0^{2n} & \text{if } \deg q = n. \end{cases}$$

# Euclidean algorithm and continued fractions

Starting from $f_0 := p$, $f_1 := q - (b_0/a_0)p$, form the Euclidean algorithm sequence

$$f_{j-1} = q_j f_j + f_{j+1}, \quad j = 1, \ldots, k, \quad f_{k+1} = 0.$$

Then $f_k$ is the greatest common divisor of $p$ and $q$. This gives a continued fraction representation

$$R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_k(z)}}}}}.$$

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# Generalized Jacobi matrices

$$\mathcal{J}(z) := \begin{bmatrix} q_k(z) & -1 & 0 & \dots & 0 & 0 \\ 1 & q_{k-1}(z) & -1 & \dots & 0 & 0 \\ 0 & 1 & q_{k-2}(z) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_2(z) & -1 \\ 0 & 0 & 0 & \dots & 1 & q_1(z) \end{bmatrix}.$$

Remark 1. $h_j(z)$ is the leading principal minor of $\mathcal{J}(z)$ of order $k-j$. In particular, $h_0(z) = \det \mathcal{J}(z)$.

Remark 2. Eigenvalues of the generalized eigenvalue problem

$$\mathcal{J}(z)u = 0$$

are closely related to the properties of $R(z)$.

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# Formulæ for Hankel minors

**Theorem.**

$$\text{If}\quad R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{q_1(z) + \cfrac{1}{q_2(z) + \cfrac{1}{q_3(z) + \cfrac{1}{\ddots + \cfrac{1}{q_k(z)}}}}},$$

with $n_j := \deg q_j$, we have, for all $j = 1, 2, \dots, k$,

$$D_{n_1+n_2+\ldots+n_j}(R) = \prod_{i=1}^{j} (-1)^{\frac{n_i(n_i-1)}{2}} \cdot (-1)^{\sum_{i=0}^{j-1} i n_{i+1}} \cdot \prod_{i=1}^{j} \frac{1}{\alpha_i^{n_i + 2\sum_{\rho=i+1}^{j} n_\rho}}.$$

# Jacobi continued fractions

In the regular case,

$$q_j(z) = \alpha_j z + \beta_j, \qquad \alpha_j, \beta_j \in \mathbb{C}, \alpha_j \neq 0.$$

The polynomials $f_j$ satisfy the three-term recurrence relation $f_{j-1}(z) = (\alpha_j z + \beta_j) f_j(z) + f_{j+1}(z), \quad j = 1, \ldots, r.$

$$R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{\alpha_1 z + \beta_1 + \cfrac{1}{\alpha_2 z + \beta_2 + \cfrac{1}{\alpha_3 z + \beta_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_r z + \beta_r}}}}}.$$

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# Stieltjes continued fractions

In the doubly regular case,

$$\begin{aligned} q_{2j}(z) &= c_{2j}, \qquad j = 1, \dots \left\lfloor \frac{k}{2} \right\rfloor, \\ q_{2j-1}(z) &= c_{2j-1}z, \qquad j = 1, \dots, r. \end{aligned}$$

$$R(z) = \frac{f_1(z)}{f_0(z)} = \cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \cfrac{1}{\ddots + \cfrac{1}{T}}}}},$$ where

$$T := \begin{cases} c_{2r} & \text{if } |R(0)| < \infty, \\ c_{2r-1}z & \text{if } R(0) = \infty. \end{cases}$$

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# Related eigenvalue problem

The associated generalized eigenvalue problem:

$$(\mathrm{A}z + \mathrm{B})u = 0,$$

$$\mathrm{A} = \begin{bmatrix} \alpha_r & 0 & \dots & 0 & 0 \\ 0 & \alpha_{r-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_2 & 0 \\ 0 & 0 & \dots & 0 & \alpha_1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} \beta_r & -1 & \dots & 0 & 0 \\ 1 & \beta_{r-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \beta_2 & -1 \\ 0 & 0 & \dots & 1 & \beta_1 \end{bmatrix}.$$

# Connections to infinite Hurwitz matrices

$$H(p,q) := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (\deg q < \deg p).$$

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# Factorization of infinite Hurwitz matrices

**Theorem.**

If $g(z) = g_0 z^l + g_1 z^{l-1} + \ldots + g_l$, then

$$H(p \cdot g, q \cdot g) = H(p, q)\mathcal{T}(g), \qquad \text{where}$$

$$\mathcal{T}(g) := \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \\ 0 & g_0 & g_1 & g_2 & g_3 & \cdots \\ 0 & 0 & g_0 & g_1 & g_2 & \cdots \\ 0 & 0 & 0 & g_0 & g_1 & \cdots \\ 0 & 0 & 0 & 0 & g_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Here we set $g_i = 0$ for all $i > l$.

# Another factorization

**Theorem.**

If the Euclidean algorithm for the pair $p, q$ is doubly regular, then $H(p,q)$ factors as

$$H(p,q) = J(c_1)\cdots J(c_k)H(0,1)\mathcal{T}(g),$$

$$J(c) := \begin{bmatrix} c & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & c & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad H(0,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

# Sturm algorithm

Sturm's algorithm is a variation of the Euclidean algorithm

$$f_{j-1}(z) = q_j(z)f_j(z) - f_{j+1}(z), \quad j = 0, 1, \ldots, k,$$

where $f_{k+1}(z) = 0$. The polynomial $f_k$ is the greatest common divisor of $p$ and $q$.
The Sturm algorithm is regular if the polynomials $q_j$ are linear.

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# Cauchy indices

**Definition.**

$$\mathrm{Ind}_\omega(F) := \begin{cases} +1, & \text{if} \quad F(\omega - 0) < 0 < F(\omega + 0), \\ -1, & \text{if} \quad F(\omega - 0) > 0 > F(\omega + 0), \end{cases}$$

is the index of the function $F$ at its real pole $\omega$ of odd order.

**Theorem [Gantmakher].**

If a rational function $R$ with exactly $r$ poles is represented by a series

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots, \qquad \text{then}$$

$$\mathrm{Ind}_{-\infty}^{+\infty} = r - 2S(D_0(R), D_1(R), D_2(R), \ldots, D_r(R)).$$

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# Stability

**Definition.**

A polynomial is stable if all its zeros lie in the left half-plane.

**Theorem.**

A polynomial $f = p(z^2) + zq(z^2)$ is stable if and only if its infinite Hurwitz matrix $H(p,q)$ is totally nonnegative.

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