## Matrix methods in stability theory

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joint work with Mikhail Tyaglov

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Dubrovnik, June 2008

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# Outline



### Complex rational functions

- Hankel and Hurwitz matrices
- Resultants and their applications
- Euclidean algorithm

### 2 Real rational functions

- Sturm algorithm
- Cauchy indices
- Stable polynomials
- Hyperbolic polynomials

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# Hankel and Hurwitz matrices

Let R(z) be a rational function expanded in its Laurent series at  $\infty$  $R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots$ .

Introduce the infinite Hankel matrix  $S := [s_{i+j}]_{i,j=0}^{\infty}$  and consider the leading principal minors of S:

$$D_{j}(S) := \det \begin{bmatrix} s_{0} & s_{1} & s_{2} & \dots & s_{j-1} \\ s_{1} & s_{2} & s_{3} & \dots & s_{j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_{j} & s_{j+1} & \dots & s_{2j-2} \end{bmatrix}, \quad j = 1, 2, 3, \dots$$

These are Hankel minors or Hankel determinants.

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### Hurwitz determinants

Let 
$$R(z) = \frac{q(z)}{p(z)}$$
,  $p(z) = a_0 z^n + \cdots + a_n$ ,  $a_0 \neq 0$ ,  
 $q(z) = b_0 z^n + \cdots + b_n$ ,

For each  $j = 1, 2, \ldots$ , denote

$$\nabla_{2j}(p,q) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{j-1} & a_j & \dots & a_{2j-1} \\ b_0 & b_1 & b_2 & \dots & b_{j-1} & b_j & \dots & b_{2j-1} \\ 0 & a_0 & a_1 & \dots & a_{j-2} & a_{j-1} & \dots & a_{2j-2} \\ 0 & b_0 & b_1 & \dots & b_{j-2} & b_{j-1} & \dots & b_{2j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_j \\ 0 & 0 & 0 & \dots & b_0 & b_1 & \dots & b_j \end{bmatrix}$$

These are the Hurwitz minors or Hurwitz determinants.

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# $Hankel \leftrightarrow Hurwitz$

### Theorem [Hurwitz].

Let R(z) = q(z)/p(z) with notation as above. Then

$$abla_{2j}(oldsymbol{
ho},oldsymbol{q})=a_0^{2j}D_j(R),\quad j=1,2,\ldots.$$

#### Corollary.

Let T(z) = -1/R(z) with notation as above. Then

$$D_j(S) = S_{-1}^{2j} D_j(T), \quad j = 1, 2, \dots$$

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## Resultant

Let *p* and *q* be as above and let  $b_0 \neq 0$ . The resultant of *p* and *q* is defined as

$$\mathbf{R}(p,q) := \det \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n & \dots & a_{2n-1} \\ 0 & a_0 & \dots & a_{n-2} & a_{n-1} & \dots & a_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_{n-1} & a_n & \dots & b_{2n-1} \\ 0 & b_0 & \dots & b_{n-2} & b_{n-1} & \dots & b_{2n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & b_n \end{bmatrix}$$

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### **Resultant formulæ**

#### Theorem.

Given polynomials *p* and *q*, let  $\lambda_i$  (*i* = 1, ..., *n*) be the zeros of *p*, and let  $\mu_j$  (*j* = 1, ..., *n*) be the zeros of *q* ( $b_0 \neq 0$ ). Then

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n-1)}{2}} \nabla_{2n}(p,q) = a_0^n \prod_{i=1}^n q(\lambda_i)$$

$$= a_0^n b_0^n \prod_{i,j=1}^n (\lambda_i - \mu_j) = (-1)^n b_0^n \prod_{j=1}^n p(\mu_j).$$

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## Resultant $\rightarrow$ discriminant

#### Definition.

Let a polynomial *p* have roots  $\lambda_i$  (*i* = 1,..., *n*). The discriminant of *p* is defined by

$$\mathbf{D}(\boldsymbol{p}) = \boldsymbol{a}_0^{2n-2} \prod_{j < i}^n (\lambda_j - \lambda_j)^2.$$

#### Theorem.

For a polynomial *p* of degree *n*,

$$\mathbf{R}(p,p') = (-1)^{\frac{n(n-1)}{2}} a_0 \mathbf{D}(p)$$

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## Orlando's formula

### Theorem [generalized Orlando].

The resultant polynomials p and q can be computed by the formula

$$\mathbf{R}(p,q) = (-1)^{\frac{n(n+1)}{2}} c \prod_{i < k} (z_i + z_k),$$

where  $z_i$  are the zeros of the polynomial  $h(z) := p(z^2) + zq(z^2)$ , and

$$c := \left\{ egin{array}{ll} a_0^{m+n} & ext{if } \deg q = m \leq n-1, \ b_0^{2n} & ext{if } \deg q = n. \end{array} 
ight.$$

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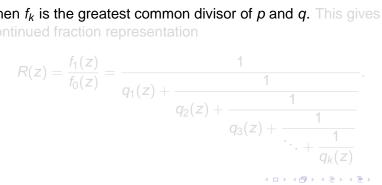
Euclidean algorithm

## Euclidean algorithm and continued fractions

Starting from  $f_0 := p$ ,  $f_1 := q - (b_0/a_0)p$ , form the Euclidean algorithm sequence

$$f_{j-1} = q_j f_j + f_{j+1}, \quad j = 1, \ldots, k, \quad f_{k+1} = 0.$$

Then  $f_k$  is the greatest common divisor of p and q. This gives a



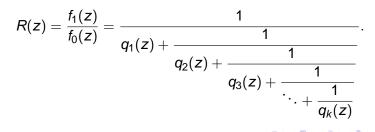
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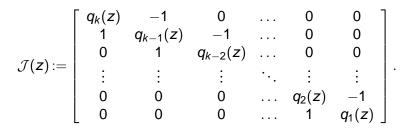
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### Generalized Jacobi matrices



**Remark 1**.  $h_j(z)$  is the leading principal minor of  $\mathcal{J}(z)$  of order k - j. In particular,  $h_0(z) = \det \mathcal{J}(z)$ .

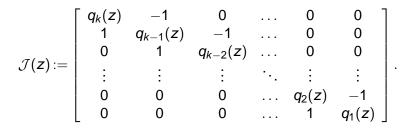
Remark 2. Eigenvalues of the generalized eigenvalue problem

$$\mathcal{J}(z)u=0$$

are closely related to the properties of R(z),

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## Generalized Jacobi matrices

$$\mathcal{J}(z) := \begin{bmatrix} q_k(z) & -1 & 0 & \dots & 0 & 0 \\ 1 & q_{k-1}(z) & -1 & \dots & 0 & 0 \\ 0 & 1 & q_{k-2}(z) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_2(z) & -1 \\ 0 & 0 & 0 & \dots & 1 & q_1(z) \end{bmatrix}$$

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## Formulæ for Hankel minors

#### Theorem.

If 
$$R(z) = rac{f_1(z)}{f_0(z)} = rac{1}{q_1(z) + rac{1}{q_2(z) + rac{1}{q_3(z) + rac{1}{rrac{1}{rrle}}}}}{rrc{1}{rrle}{rrle}}}}{rrle}{rrle}{rrle}{1}} }}}}}}}}}}}}}}}}}}}}}}}}}}}}}}{r{{rrle}{rrle}{1}}{rrle}{rrle}{rrle}{1}{rrle}}{rrle}{rrle}{1}{rrle}{rrle}}{rrle}{rrle}{rrle}{1}{rrle}}{rrle}}{rrle}{1}{rrle}}{rrle}{1}{rrle}{1}{rrle}}{rrle}{1}{rrle}}{rrle}{}}{rrle}{1}{rrle}{1}{rrle}}{1}{rrle}}{1}{rrle}{1}{rrle}}{rrle}{1}{rrle}{1}{rrle}{1}{rrle}{1}{rrle}{1}{rrle}{1}{rrle}{1}{rrle}{1}{$$

with  $n_j := \deg q_j$ , we have, for all  $j = 1, 2, \ldots, k$ ,

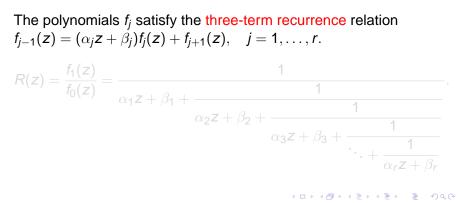
$$D_{n_1+n_2+\ldots+n_j}(R) = \prod_{i=1}^{j} (-1)^{\frac{n_j(n_j-1)}{2}} \cdot (-1)^{\sum_{i=0}^{j-1} i n_{i+1}} \cdot \prod_{i=1}^{j} \frac{1}{\alpha_j^{n_i+2\sum_{\rho=i+1}^{j} n_\rho}}.$$

Euclidean algorithm

## Jacobi continued fractions

In the regular case,

$$q_j(z) = \alpha_j z + \beta_j, \qquad \alpha_j, \beta_j \in \mathbb{C}, \alpha_j \neq 0.$$



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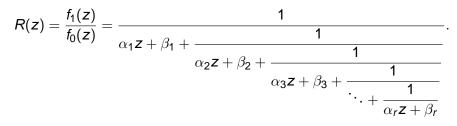
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The polynomials  $f_j$  satisfy the three-term recurrence relation  $f_{j-1}(z) = (\alpha_j z + \beta_j) f_j(z) + f_{j+1}(z), \quad j = 1, ..., r.$ 



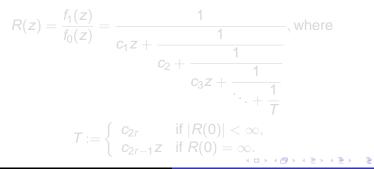
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## Stieltjes continued fractions

In the doubly regular case,

$$\begin{array}{rcl} q_{2j}(z) & = & c_{2j}, & j=1,\ldots \left\lfloor \frac{k}{2} \right\rfloor, \\ q_{2j-1}(z) & = & c_{2j-1}z, & j=1,\ldots,r. \end{array}$$

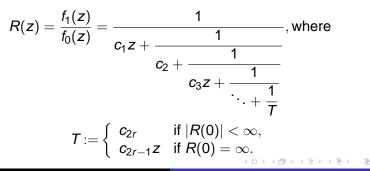


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## Related eigenvalue problem

The associated generalized eigenvalue problem:

 $(\mathbf{A}\mathbf{z}+\mathbf{B})\mathbf{u}=\mathbf{0},$ 

$$\mathbf{A} = \begin{bmatrix} \alpha_r & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha_{r-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \alpha_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \alpha_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \beta_r & -\mathbf{1} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \beta_{r-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \beta_2 & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} & \beta_1 \end{bmatrix}$$

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Connections to infinite Hurwitz matrices

$$H(p,q) := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} (\deg q < \deg p).$$

$$H(p,q) = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} (\deg q = \deg p).$$

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# Factorization of infinite Hurwitz matrices

#### Theorem.

If  $g(z) = g_0 z' + g_1 z'^{-1} + \ldots + g_l$ , then

 $H(p \cdot g, q \cdot g) = H(p, q)T(g),$  where

$$\mathcal{T}(g) := egin{bmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \ 0 & g_0 & g_1 & g_2 & g_3 & \cdots \ 0 & 0 & g_0 & g_1 & g_2 & \cdots \ 0 & 0 & 0 & g_0 & g_1 & \cdots \ 0 & 0 & 0 & g_0 & g_1 & \cdots \ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \ \end{pmatrix}$$

Here we set  $g_i = 0$  for all i > I.

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# Another factorization

#### Theorem.

If the Euclidean algorithm for the pair p, q is doubly regular, then H(p, q) factors as

$$H(p,q) = J(c_1) \cdots J(c_k)H(0,1)\mathcal{T}(g),$$

$$J(c) := \begin{bmatrix} c & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & c & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} H(0,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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# Sturm algorithm

Sturm's algorithm is a variation of the Euclidean algorithm

$$f_{j-1}(z) = q_j(z)f_j(z) - f_{j+1}(z), \quad j = 0, 1, \dots, k,$$

where  $f_{k+1}(z) = 0$ . The polynomial  $f_k$  is the greatest common divisor of p and q. The Sturm algorithm is regular if the polynomials  $q_i$  are linear.

The Sturm algorithm was originally proposed to count zeros on a real interval.

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Sturm algorithm Cauchy indices Stable polynomials Hyperbolic polynomials

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Sturm algorithm Cauchy indices Stable polynomials Hyperbolic polynomials

# Cauchy indices

### Definition.

$$\operatorname{Ind}_{\omega}(F) := \left\{ egin{array}{ccc} +1, & ext{if} & F(\omega-0) < 0 < F(\omega+0), \ -1, & ext{if} & F(\omega-0) > 0 > F(\omega+0), \end{array} 
ight.$$

is the index of the function *F* at its real pole  $\omega$  of odd order.

#### Theorem [Gantmakher].

If a rational function *R* with exactly *r* poles is represented by a series

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots$$
, then

 ${\sf Ind}_{-\infty}^{+\infty} = r - 2S(D_0(R), D_1(R), D_2(R), \dots, D_r(R)).$ 

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# Cauchy indices

### Definition.

$$\operatorname{Ind}_{\omega}(F) := \left\{ egin{array}{ccc} +1, & ext{if} & F(\omega-0) < 0 < F(\omega+0), \ -1, & ext{if} & F(\omega-0) > 0 > F(\omega+0), \end{array} 
ight.$$

is the index of the function *F* at its real pole  $\omega$  of odd order.

#### Theorem [Gantmakher].

If a rational function *R* with exactly *r* poles is represented by a series

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# Stability

### Definition.

A polynomial is stable if all its zeros lie in the left half-plane.

#### Theorem.

A polynomial  $f = p(z^2) + zq(z^2)$  is stable if and only if its infinite Hurwitz matrix H(p, q) is totally nonnegative.

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