Modal Approximations to Damped Systems

Krešimir Veselić, Fernuniversität Hagen

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$$\text{System:} \quad M\ddot{x} + C\dot{x} + Kx = 0, \tag{1}$$

M, C, K real symm. matrices with M, K positive definite and C positive semidefinite. The *phase space formulation* of (1):

$$\dot{y} = Ay,$$

$$A = \begin{bmatrix} 0 & L_1^T L_2^{-T} \\ -L_2^{-1} L_1 & -L_2^{-T} C L_2^{-1} \end{bmatrix},$$

$$y = \begin{bmatrix} L_1^T x \\ L_2^T \dot{x} \end{bmatrix}, \quad K = L_1 L_1^T,\ M = L_2 L_2^T.$$

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The total energy identity

$$\|y\|^2 = \dot{x}^T M \dot{x} + x^T K x.$$

The quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)x = 0 \tag{2}$$

is equivalent to $Ay = \lambda y$.
Spectral decomposition of A, needed for e^{At}, may not exist at all, even for physically relevant systems.

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$$\Phi^T K \Phi = \Omega^2 = diag(\omega_1^2, \ldots, \omega_n^2), \quad \Phi^T M \Phi = I$$

$\omega_1 \leq \cdots \leq \omega_n$ are *the undamped frequencies*. Obtain

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Simultaneaus diagonalizability $\Longleftrightarrow$ D commutes with Ω.

Instead of merely proportional damping we take any block-diagonal part D^0 of D that commutes with Ω (generically just the diagonal part of D) This is called a modal approximation. Its eigenvalues are

$$\lambda_{\pm}^{j} = \frac{-d_{jj} \pm \sqrt{d_{jj}^2 - 4\omega_j^2}}{2}.$$

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Start with perturbation bounds. They will comprise

- Undamped approximation (as a prelude)
- Modal approximations

Both are performed by

- Spectral norm bounds
- Gershgorin type bounds

Undamped approximation.

In the phase space we have $A = A_0 + B$:

$$A_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.$$

A_0 is skew-symmetric, so by standard perturbation theory $\sigma(A)$ is contained in the union of disks of radius

$$\|D\| = \max \frac{x^T C x}{x^T M x}$$

centred at $\pm i\omega_j$.

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centred at $\pm i\omega_j$. Too crude an estimate — B has so many zeros.

Remedy: Turn back to the quadratic eigenvalue formulation. The inverse

$$(\lambda^2 I + \lambda D + \Omega^2)^{-1} =$$

$$(\lambda^2 I + \Omega^2)^{-1}(I + \lambda D(\lambda^2 I + \Omega^2)^{-1})^{-1}$$

exists, if

$$\|(\lambda^2 I + \Omega^2)^{-1}\|\|D\| = \max_j \frac{\|D\|}{|\lambda - i\omega_j||\lambda + i\omega_j|} < 1$$

Hence the bound:

$$\sigma(A) \subseteq \cup_j \mathcal{C}(i\omega_j, -i\omega_j, \|D\|).$$

Here

$$\mathcal{C}(\lambda_+, \lambda_-, r) = \{\lambda : |\lambda - \lambda_+||\lambda - \lambda_-| \le |\lambda| r\}$$

are *stretched Cassini* ovals with foci $\lambda_\pm$ and extension r. May consist of one or two components; the latter when r is small with respect to $|\lambda_+ - \lambda_-| = 2\omega_j$. Then

$$\mathcal{C}(\lambda_+, \lambda_-, r) \approx \left\{\lambda : |\lambda \pm i\omega_j| \le \frac{\|D\|}{2}\right\} \tag{3}$$

disks, one half of the standard radius! Purely first order estimate. Requires no separation of the undamped eigenvalues.

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disks, one half of the standard radius! Purely first order estimate. Requires no separation of the undamped eigenvalues. Gershgorin-type estimate immediate

$$\sigma(A) \subseteq \cup_j \mathcal{C}(i\omega_j, -i\omega_j, R_j), \quad R_j = \sum_{k=1}^{n} |d_{jk}|.$$

Modal approximation.

The standard methods in the phase space get clumsy. The quadratic eigenvalue approach stays elegant: $D = D^0 + D'$, $D^0 = \text{diag}(\text{diag}(D))$ and

$$(\lambda^2 I + \lambda D + \Omega^2)^{-1}$$

exists, if

$$\frac{|\lambda| \|D'\|}{\min_j(|\lambda - \lambda_+^j||\lambda - \lambda_-^j|)} < 1$$

where $\lambda_\pm^j$ are given by

$$(\lambda - \lambda_+^j)(\lambda - \lambda_-^j) = \lambda^2 + \lambda d_{jj} + \omega_j^2$$

Hence

$$\sigma(A) \subseteq \cup_j \mathcal{C}(\lambda_+^j, \lambda_-^j, \|D'\|), \tag{4}$$

Did we obtain tighter bound than those with undamped approximation? Is $\|D'\|$ always smaller than $\|D\|$? Well, yes, and more.

Theorem. Let $D = D^*$ and D' any its block diagonal part. Then

$$\|D'\| \leq \mathrm{spread}(D)$$

If, D is pos. semidefinite then

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Hence: the coarser block diagonal part is extracted, the smaller norm (and better eigenvalue bound) is obtained. This requires multiplicities among ω_j (tight clusterings would also do).

Gershgorin bounds also immediate.

$$\sigma(A) \subseteq \cup_j \mathcal{C}(\lambda_+^j, \lambda_-^j, r_j)$$

with

$$r_j = \sum_{\substack{k=1 \\ k \neq j}}^{p} |d_{jk}|.$$

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Long existing phase space Gershgorin disk bound (P. Lancaster)

$$\sigma(A) \subseteq \cup_j \{\lambda : |\lambda \mp i\omega_j + \frac{d_{jj}}{2}| \leq r_j'\}$$

with

$$r_j' = r_j + \frac{|d_{jj}|}{2}$$

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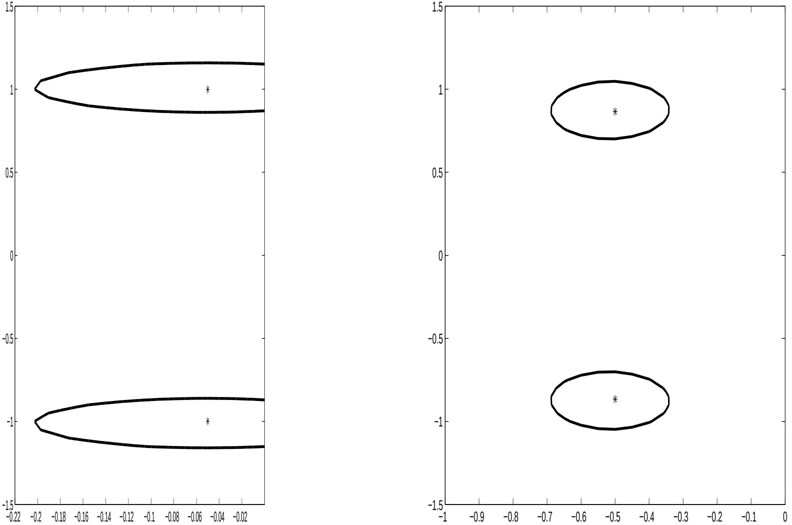
Relevant only for small d_{jj}, but then our ovals are about twice as narrow as the circles above. Need: tight bounds for the diameters of stretched ovals.

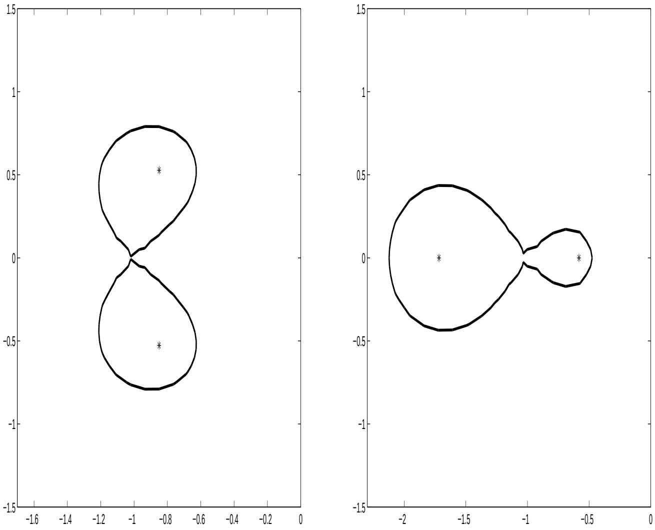
Coarser block estimates also possible, when allowed by multiple ω_j.
Often, but not always, better bounds.

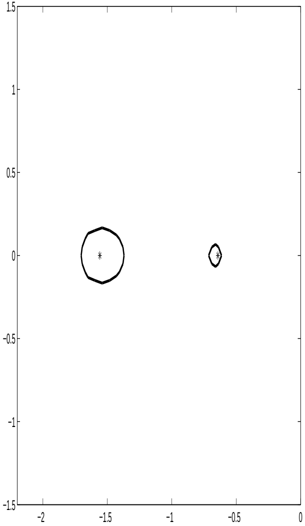
We plot some stretched ovals.

Ovals for $\omega = 1; d = 0.1, 1; \ r = 0.3$:



Ovals for $\omega = 1$; $d = 1.7, 2.3, 2.2$; $r = 0.3, 0.3, 0.1$:




1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0

Brauer-like ovals can also be incorporated. The spectrum is contained in the union of *double ovals*

$$\mathcal{D}(\lambda_+^p, \lambda_-^p, \lambda_+^q, \lambda_-^q, r_p r_q) =$$

$$\{\lambda : |\lambda - \lambda_+^p||\lambda - \lambda_-^p||\lambda - \lambda_+^q||\lambda - \lambda_-^q| \leq r_p r_q |\lambda|^2\},$$

where the union is taken over all pairs $p \neq q$ and $\lambda_\pm^p$ are the solutions of $\lambda^2 + d_{pp}\lambda + \omega_p^2 = 0$ and similarly for $\lambda_\pm^q$.

A bound for the matrix exponential

Recall: $D = D^0 + D'$. Then

$$|x^T D' y|^2 \le \epsilon^2 x^T D^0 x y^T D y,$$

for any $\epsilon > 0$ (sic!) Then

$$\|e^{At} - e^{A^0 t}\| \le \frac{\varepsilon}{2}.$$

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is the modal part of A.
Allows to obtain bounds for the exponential decay of $\|e^{At}\|$.

The computation or simple estimation of 2×2 exponential is not quite trivial (ask Beresford). We plot the norm of

$$exp\left(\begin{bmatrix} 0 & 1 \\ -1 & -2\theta \end{bmatrix} \tau\right)$$

for various θ and the 'absolute time' $\tau \in [1, 6]$: