A Quadratically Constrained Eigenvalue Minimization Problem Arising from PDE of Monge-Ampère Type

D.C. Sorensen

- Collaborator: R. Glowinski
- Support: NSF and AFOSR

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Fooling Around and Having Fun with Numerical Linear Algebra

Outline

- Motivation — Monge-Ampère Equations
- FEM Discretization $\longrightarrow$ Quadratic Min Problem
- Formulation of General Eigenvalue Constrained Quadratic Min Problem
- The General Secular Equation and Its Solution
- A Surprising Result — Finite $\mathcal{O}(n^3)$ Complexity
- Some Numerical Results

Monge-Ampère Equations

Applications and Research Reference:

BIRS Workshop (2003) on Monge-Ampère Equation
http://www.birs.ca/workshops/2003/03w5067/report03w5067.pdf

Monge-Ampère Equations arise in

Riemannian Geometry

Conformal Geometry

CR Geometry

Example : Problem of prescribed Gauss curvature

A real-valued function K is specified on a domain Ω in $\mathbb{R}^n$
Prescribed Gauss curvature seeks to identify a hypersurface of $\mathbb{R}^{n+1}$ as a graph $z = u(x)$ over $x \in \Omega$ so that the Gauss curvature is given by $K(x)$ at every x.

Monge-Ampère Problem

Fully-Nonlinear 3D BVP (of Dirichlet type);

Find ψ such that

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = f \text{ in } \Omega, \;\; \psi = g \text{ on } \partial\Omega,$$

where

- Function ψ unknown; Ω is a bounded domain of $\mathbb{R}^3$
- $\{\lambda_1, \lambda_2, \lambda_3\}$ is the spectrum of

  Hessian $\mathbf{D}^2\psi = \left(\frac{\partial^2\psi}{\partial x_i, \partial x_j}\right)_{1\leq i,j\leq 3}$
- f and g are two given functions with $f > 0$.

Dirichlet Problem for σ_2-Operator

Rewrite as

$$|\nabla^2\psi|^2 - \mathbf{D}^2\psi : \mathbf{D}^2\psi = 2f \text{ in } \Omega,$$

where

$$\mathbf{A} : \mathbf{B} = \sum_{1\leq i,j\leq d} a_{ij}b_{ij} = \mathrm{trace}\mathbf{A}^T\mathbf{B} \quad (\textit{Frobenius})$$

Fully nonlinear PDE becomes

$$[\mathrm{trace}\{\mathbf{D}^2\psi\}]^2 - \mathrm{trace}\{(\mathbf{D}^2\psi)^2\} = 2f.$$

Solving when σ_2-Operator is Elliptic

Least Squares Approach:

σ_2-operator linearized in neighborhood of elliptic ψ gives:

$$\phi \to 2[\nabla^2\psi\nabla^2\phi - \mathbf{D}^2\psi : \mathbf{D}^2\phi].$$

Coefficient Matrix

$$2[\nabla^2\psi\mathbf{I} - \mathbf{D}^2\psi],$$

The σ_2-operator *elliptic* in nbhd of $\psi \iff$ if Matrix is s.p.d. (or n.p.d) , i.,e.,

$$(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3) > 0, (\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2) > 0.$$

$$\lambda_1 + \lambda_2 > 0, \quad \lambda_2 + \lambda_3 > 0, \quad \lambda_1 + \lambda_3 > 0.$$

PDE Least Squares Problem

Find $(\psi, \mathbf{P}) \in \mathcal{V}_g \times \mathbf{Q}_f$ such that
$J(\psi, \mathbf{P}) \leq J(\phi, \mathbf{G}), \quad \forall (\phi, \mathbf{G}) \in \mathcal{V}_g \times \mathbf{Q}_f,$
where

$$J(\phi, \mathbf{G}) = \frac{1}{2} \int_\Omega (\mathbf{D}^2 \phi - \mathbf{G}) : (\mathbf{D}^2 \phi - \mathbf{G}) \mathrm{d}x,$$

with $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$.

Block Relaxation Algorithm

Given $\psi^0 \in \mathcal{V}_g$;
for $k = 0, 1, 2, \ldots$

- $\mathbf{P}^{k+1} = argmin_{\mathbf{G} \in \mathbf{Q}_f} J(\psi^k, \mathbf{G})$;
- $\psi^{k+1/2} = argmin_{\phi \in \mathcal{V}_g} J(\phi, \mathbf{P}^{k+1})$;
- $\psi^{k+1} = \psi^k + \omega(\psi^{k+1/2} - \psi^k)$;

Relaxation Parameter: $0 < \omega < 2$
Initialization

$$\nabla^2 \psi^0 = \sqrt{3f} \;\; \text{in} \;\; \Omega, \quad \psi^0 \text{on } \partial\Omega.$$

Note that ψ^0 has the $\mathcal{H}^2(\Omega)$-regularity if $\partial\Omega$ is 'sufficiently' smooth and/or Ω is convex.

Multiple Minimizations Needed

Must solve following at the vertices of a finite element mesh:

$$\text{Find} \quad \mathbf{P}^{k+1}(x) \in \mathbf{E}(x), \quad \mathbf{j}_k(\mathbf{P}^{k+1}(x); x) \leq \mathbf{j}_k(\mathbf{A}; x), \quad \forall \mathbf{A} \in \mathbf{E}(x),$$

where

$$\begin{aligned} \mathbf{E}(x) = \{\mathbf{A} | \mathbf{A} \in \mathbb{R}^{3\times3}, \quad \mathbf{A} = \mathbf{A}^T, \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = f(x), \\ \lambda_1 + \lambda_3 > 0, \lambda_2 + \lambda_3 > 0, \lambda_3 + \lambda_1 > 0\} \end{aligned}$$

and

$$\mathbf{j}_k(\mathbf{A}; x) = \frac{1}{2}\mathbf{A} : \mathbf{A} - \mathbf{D}^2\psi^k(x) : \mathbf{A},$$

with $\{\lambda_1, \lambda_2, \lambda_3\}$ being the spectrum of $\mathbf{A}$.

Multiple Minimizations Needed

Must solve following at the vertices of a finite element mesh: Can normalize using division by $f(x)$.

$$\min_{\mathbf{A}\in\mathbf{E}_1} \mathrm{trace}[\mathbf{A}^T(\mathbf{A}-2\mathbf{B})] = \min_{\mathbf{A}\in\mathbf{E}_1} \mathrm{trace}[\mathbf{A}^2-2\mathbf{AB})]$$

where

$$\begin{aligned}\mathbf{E}_1 = \{\mathbf{A}|\mathbf{A} \in \mathbb{R}^{3\times 3}, \mathbf{A} = \mathbf{A}^T,\\ \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = 1\ ,\\ \lambda_1+\lambda_3>0, \lambda_2+\lambda_3>0, \lambda_3+\lambda_1>0\}\end{aligned}$$

(with $\{\lambda_1, \lambda_2, \lambda_3\}$ being the spectrum of $\mathbf{A}$)

Minimization Problem - Matrix Form

Problem Qmin

$$
\begin{aligned}
&\min \operatorname{trace}\{\mathbf{AA} - 2\mathbf{BA}\} \\
&\text{s.t.} \\
&\ell^T \mathbf{M} \ell = 2 \\
&\mathbf{M}\ell \geq \mathbf{0}
\end{aligned}
$$

where

$$\mathbf{M} = \mathbf{e}\mathbf{e}^T - \mathbf{I}, \quad \text{with} \quad \mathbf{e}^T = (1, 1, \ldots, 1),$$

and

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}^T \quad \text{is specified}, \\
\mathbf{A} &= \mathbf{A}^T = \mathbf{Q}\Lambda\mathbf{Q}^T, \\
\Lambda &= \operatorname{diag}(\ell).
\end{aligned}
$$

Diagonalized Problem

Note that Problem Qmin is equivalent to

$$\min \operatorname{trace}\{\Lambda\Lambda - 2\hat{\mathbf{B}}\Lambda\} = \min \ell^T\ell - 2\mathbf{b}^T\ell$$
$$\text{s.t.}$$
$$\ell^T\mathbf{M}\ell = 2$$
$$\mathbf{M}\ell \geq \mathbf{0}$$

where

$$\hat{\mathbf{B}} = \mathbf{Q}^T\mathbf{B}\mathbf{Q} \quad \text{and} \quad \mathbf{b} = \operatorname{diag}(\hat{\mathbf{B}}).$$

Constraints for $n = 3$

Constraints:

$$\ell^T \mathbf{M} \ell = 2$$
$$\mathbf{M} \ell \geq \mathbf{0}$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\ell^T \mathbf{M} \ell = 2 \quad \Rightarrow \quad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$$

$$\mathbf{M} \ell \geq \mathbf{0} \quad \Rightarrow \quad \begin{array}{l} \lambda_2 + \lambda_3 \geq 0 \\ \lambda_1 + \lambda_3 \geq 0 \\ \lambda_1 + \lambda_2 \geq 0 \end{array}$$

No Active Equality Consraint

Lemma

If the vector $\ell \in \mathbb{R}^n$ is finite and feasible, then none of the inequality constraints can be active. In other words,

$$\ell^T \mathbf{M} \ell = 2 \quad \Rightarrow \quad \mathbf{e}_j^T \mathbf{M} \ell > 0, \text{ for } j = 1, 2, \ldots, n.$$

Proof Outline

Proof Outline:

If $\lambda_2 + \lambda_3 + \cdots + \lambda_n = 0$,

Equality constraint $\ell^T \mathbf{M} \ell = 2$ provides

$$\begin{aligned}
\lambda_1(\lambda_2 + \lambda_3 + \cdots + \lambda_n) = 2 \quad & - \quad \lambda_2(\lambda_3 + \lambda_4 + \ldots \lambda_n) \\
& - \quad \lambda_3(\lambda_2 + \lambda_4 + \lambda_5 + \ldots \lambda_n) \\
& \ldots \\
& - \quad \lambda_n(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}).
\end{aligned}$$

It follows that

$$0 = \lambda_1(\lambda_2 + \lambda_3 + \cdots + \lambda_n) = 2 + \lambda_2^2 + \lambda_3^2 \cdots + \lambda_n^2 \geq 2$$

which is a contradiction.

Lagrangian for Equality Constraint:

Lagrangian:

$$\mathcal{L}(\ell, \mu) = \ell^T \ell - 2\mathbf{b}^T \ell + \mu(\ell^T \mathbf{M} \ell - 2).$$

Setting the grad of Lagrangian to zero gives:

$$(\mathbf{I} + \mu\mathbf{M})\ell = \mathbf{b}.$$

If $1/\mu$ is not an eigenvalue of $-\mathbf{M}$ then the equality constraint becomes

$$\mathbf{b}^T(\mathbf{I} + \mu\mathbf{M})^{-1}\mathbf{M}(\mathbf{I} + \mu\mathbf{M})^{-1}\mathbf{b} = 2$$

Eigensystem of $\mathbf{M}$

Helps to know eigensystem of $\mathbf{M} = \mathbf{e}\mathbf{e}^T - \mathbf{I}$:

Eigenvalues

$$\omega_1 = n - 1 \text{ and } \omega_2 = -1, \text{ multiplicity } n - 1$$

Eigenvector for ω_1 is $\mathbf{e}$

Eigenvector Matrix

$$\mathbf{U} \equiv (\mathbf{I} - 2\mathbf{w}\mathbf{w}^T), \text{ with } \mathbf{w} = (\mathbf{e} + \sqrt{n}\mathbf{e}_1)/\|\mathbf{e} + \sqrt{n}\mathbf{e}_1\|,$$

Easily checked:

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{U}\mathbf{M}\mathbf{U}^T = \mathbf{U}\mathbf{e}\mathbf{e}^T\mathbf{U}^T - \mathbf{I} = n\mathbf{e}_1\mathbf{e}_1^T - \mathbf{I}.$$

The Secular Equation

$$\frac{\beta_1^2 \omega_1}{(1+\mu\omega_1)^2} = 2 + \frac{\beta_2^2}{(1-\mu)^2}$$

where $(\beta_1, \mathbf{b}_2^T) = \mathbf{b}^T \mathbf{U}$ and $\beta_2^2 = \mathbf{b}_2^T \mathbf{b}_2$.

Note:

$\beta_1 = \mathbf{e}^t \mathbf{b}/\sqrt{n}$ is *invariant*: $\quad \mathbf{e}^T \mathbf{b} = \mathrm{trace}\{\mathbf{B}\}$

Reciprocal Square Root Equation

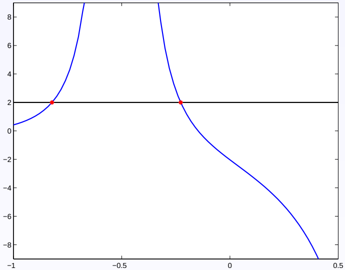
Better Equivalent Form

$$\pm(1+\mu\omega_1) = \frac{(1-\mu)|\beta_1|\sqrt{\omega_1}}{\sqrt{2(1-\mu)^2+\beta_2^2}}.$$

Algorithm: Newton's Method

converges in 3 steps

Graphs of Secular Equations



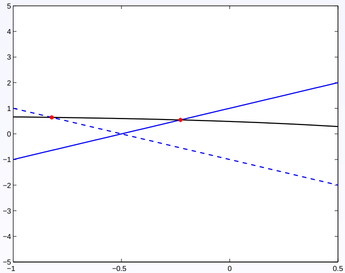


Figure: The Secular Equation for Multiplier μ (left) and the Reciprocal Square Root Secular Equation (right)

Choice of Root

Corresponding to Root μ, the vector ℓ is:

$$\ell = \mathbf{U}\mathbf{c} \quad \text{with} \quad \mathbf{c}^T = (\frac{\beta_1}{(1+\mu\omega_1)}, \quad \frac{1}{1-\mu}\mathbf{b}_2^T).$$

Pick Solution with Positive Components:

$$0 < \mathbf{M}\ell = (\mathbf{e}\mathbf{e}^T - \mathbf{I})\ell = \mathbf{e}(\mathbf{e}^T\ell) - \ell$$

Suggested Alternating Min Algorithm

- $\mathbf{b} = \mathrm{diag}(\mathbf{B})$; $\mathbf{Q} = \mathbf{I}$;
- while ('not converged'),
  - $\min_{\ell} \ell^T \ell - 2\mathbf{b}^T \ell$ s.t. constraints ;
  - $\min_{\mathbf{A}=\mathbf{W}\mathrm{diag}(\ell)\mathbf{W}^T} \mathrm{trace}\{\mathbf{AA} - 2\mathbf{BA}\}$ s.t. constraints;
  - $\mathbf{b} = \mathrm{diag}(\mathbf{W}^T\mathbf{BW})$;

ALWAYS CONVERGED IN 2 STEPS !

Surprise Result

Suppose $\mathbf{B} = \mathbf{Q}\mathrm{diag}(\mathbf{b})\mathbf{Q}^T$
If μ and corresponding $\ell = (\mathbf{I} + \mu\mathbf{M})^{-1}\mathbf{b}$ solve

$$\min \ell^T \ell - 2\mathbf{b}^T \ell \quad \text{s.t.constraints}$$

We have

Lemma
Let $\Lambda = \mathrm{diag}(\ell)$. *Then* $\mathbf{A_Q} = \mathbf{Q}\Lambda\mathbf{Q}^T$ *solves*

$$\min \mathrm{trace}\{\mathbf{AA} - 2\mathbf{BA}\} \quad \text{s.t.constraints}$$

over all $\mathbf{A} = \mathbf{W}\Lambda\mathbf{W}^T$ *with* $\mathbf{W}^T\mathbf{W} = \mathbf{I}$.

Proof Outline

If $\hat{\mathbf{b}} = \mathbf{Q}^T\mathbf{B}\mathbf{Q}$ then $\mathbf{W}^T\mathbf{B}\mathbf{W} = \hat{\mathbf{Q}}^T\hat{\mathbf{B}}\hat{\mathbf{Q}}$.
Let $\hat{\mathbf{b}} = \mathrm{diag}(\hat{\mathbf{Q}}^T\hat{\mathbf{B}}\hat{\mathbf{Q}})$.

$$\hat{\beta}_j \equiv \hat{\mathbf{b}}(j) = \mathbf{q}_j^T\hat{\mathbf{B}}\mathbf{q}_j = \sum_{i=1}^{n} \beta_i\gamma_{ij}^2,$$

where $\beta_j = \mathbf{b}(j)$, $\hat{\mathbf{Q}} = (\gamma_{ij})$, $\mathbf{q}_j = \hat{\mathbf{Q}}\mathbf{e}_j$
Hence,

$$\hat{\mathbf{b}} = \mathbf{G}^T\mathbf{b}, \quad \text{where the } i,j - \text{th entry of } \mathbf{G} \ \text{ is } \gamma_{ij}^2.$$

We show

$$\ell^T\ell - 2\mathbf{b}^T\ell \leq \ell^T\ell - 2\hat{\mathbf{b}}^T\ell.$$

Since ℓ is fixed, sufficient to show

$$\ell^T\hat{\mathbf{b}} - \ell^T\mathbf{b} = \ell^T(\hat{\mathbf{b}} - \mathbf{b}) \leq 0.$$

Proof Outline Contd.

$\mathbf{Ge} = \mathbf{G}^T\mathbf{e} = \mathbf{e}$ implies

$$\hat{\mathbf{b}} - \mathbf{b} = (1-\mu)(\mathbf{G}^T - \mathbf{I})\ell,$$

Therefore,

$$\ell^T(\hat{\mathbf{b}} - \mathbf{b}) = (1-\mu)\frac{1}{2}(\ell^T(\mathbf{G}^T - \mathbf{I})\ell + \ell^T(\mathbf{G} - \mathbf{I})\ell)$$

It follows that

$$\ell^T(\hat{\mathbf{b}} - \mathbf{b}) = -(1-\mu)\sum_{i \neq j} \gamma_{ij}^2(\lambda_j - \lambda_i)^2 \leq 0,$$

since $(1-\mu) > 0$.

Converse

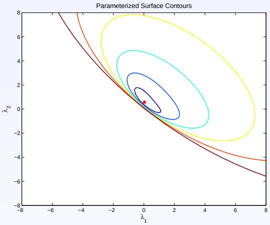
Lemma

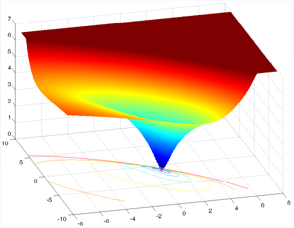
Suppose $\mathbf{A} = \mathbf{W}\Lambda\mathbf{W}^T$ *solves Problem Qmin. Then there is an orthogonal* $\hat{\mathbf{Q}}$ *such that* $\mathbf{Q} = \mathbf{W}\hat{\mathbf{Q}}$ *diagonalizes* $\mathbf{B}$ *and* $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$*. In other words, if* $\mathbf{A}$ *solves Problem Qmin, then* $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ *with* $\mathbf{Q}^T\mathbf{B}\mathbf{Q}$ *diagonal.*

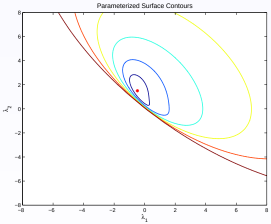
Final Algorithm

- $[\mathbf{Q}, \mathbf{L}] = \mathrm{eig}(\mathbf{B})$; $\mathbf{b} = \mathrm{diag}(\mathbf{L})$
- $\min_\ell \ell^T \ell - 2\mathbf{b}^T \ell$ subject to constraints
- $\mathbf{A} = \mathbf{Q}\mathrm{diag}(\ell)\mathbf{Q}^T$;

Works for ANY n


Parameterized Surface Contours
λ_1
λ_2




Parameterized Surface Contours
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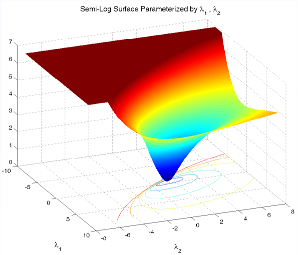
Semi-Log Surface Parameterized by λ_1 , λ_2
λ_1
λ_2

Summary

- Monge-Ampère important to Diff Geometry
- Glowinski - Dean algorithm requires MANY eigenvalue constrained min problems
- We provided a very simple and efficient method with some surprising properties.

Report:

A Quadratically Constrained Minimization Problem Arising from PDE of Monge-Ampère Type

CAAM TR08-02, DCS and R. Glowinski

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