Numerical solution of eigenvalue problems from acoustic field computations

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DFG Research Center MATHEON

Mathematics for key technologies

Overview

- Introduction.
- Application.
- Linear system.
- Eigenvalue problems.
- Future challenges.
- Conclusions.
- A cry for help.

The analysis of the acoustic behavior of structures and vehicles needs the numerical solution of parameter dependent linear systems and eigenvalue problems.

- ▷ Such systems have been solved for decades!
- ▷ The mathematics is well-known and used in industrial engineering every day!
- ▷ Numerical methods are available in (commercial) software! (NASTRAN)
- ▷ Do we still need to talk about it?
- ▷ Do we need improved numerical methods?
- ▷ Is the achieved accuracy acceptable?
- ▷ **What are the challenges?**

Optimality through mathematics

- Society is increasingly sensitive to inconveniences that come with modern technologies such as air and water pollution, noise by airplanes, cars, trains.
- There is an increasing demand for optimal solutions. Minimal energy consumption, minimal noise, pollution, waste.
- Optimal solutions are obtained by using mathematical techniques, such as model based optimization/ optimal control.
- We need better mathematical models, faster and more accurate numerical methods, robust implementations on modern computer architectures.
- Industrial problems create interesting new mathematical problems.
- Discretization methods, optimization methods and numerical linear algebra methods must go hand in hand.

Acoustic field in car interior

Project with company SFE in Berlin 2007/2008.

- Computation of acoustic field for coupled system of car body and air.
- Use of SFEs parameterized FEM model which allows geometry and topology changes.
- Frequent solution of linear systems and eigenvalue problems (up to size $10,000,000$) within optimization loop that changes geometry, topology, damping material, etc.
- Ultimate goal: Minimize noise in important regions in car interior.

The project

- Numerical methods for large scale structured parameter dependent linear systems.
- These methods are used to determine the frequency response of the system under excitations.
- Numerical methods for large scale structured parameter dependent nonlinear eigenvalue problems (model reduction for coupled model), modal analysis, optimization of frequencies.
- Determine all eigenvalues in a given region of $\mathbb{C}$.
- Determine projectors on important spectral subspaces for model reduction.
- Implementation of parallel solver in SFE Concept.

SFE
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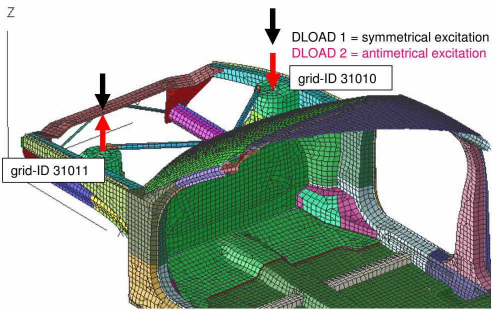
SFE AKUSMOD

FE Model: Excitation

Unit force = 1 N mm



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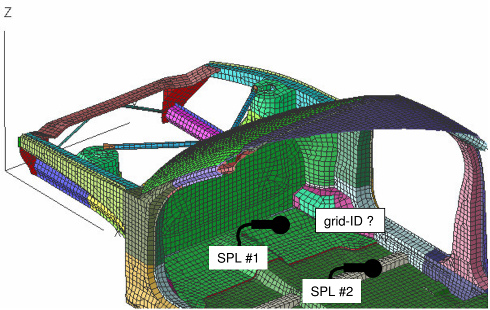
Frequency response II

SFE AKUSMOD

FE Model: SPL for two microphone positions



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Mathematical model: Linear system

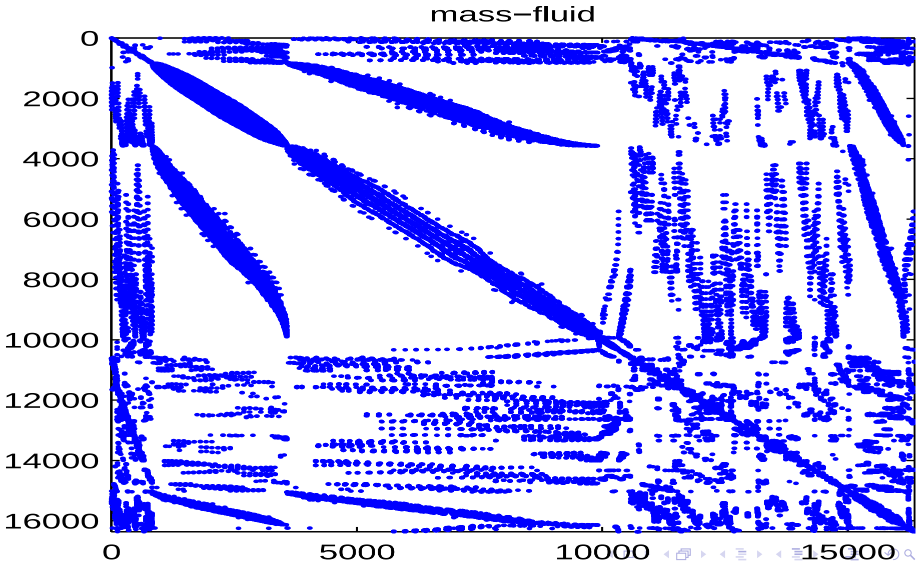
Solve $P(\omega, \alpha)u(\omega, \alpha) = f(\omega, \alpha)$, where

$$P(\omega, \alpha) := -\omega^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \imath\omega \begin{bmatrix} D_s & D_{as}^T \\ D_{as} & D_f \end{bmatrix} + \begin{bmatrix} K_s(\omega) & 0 \\ 0 & K_f \end{bmatrix},$$

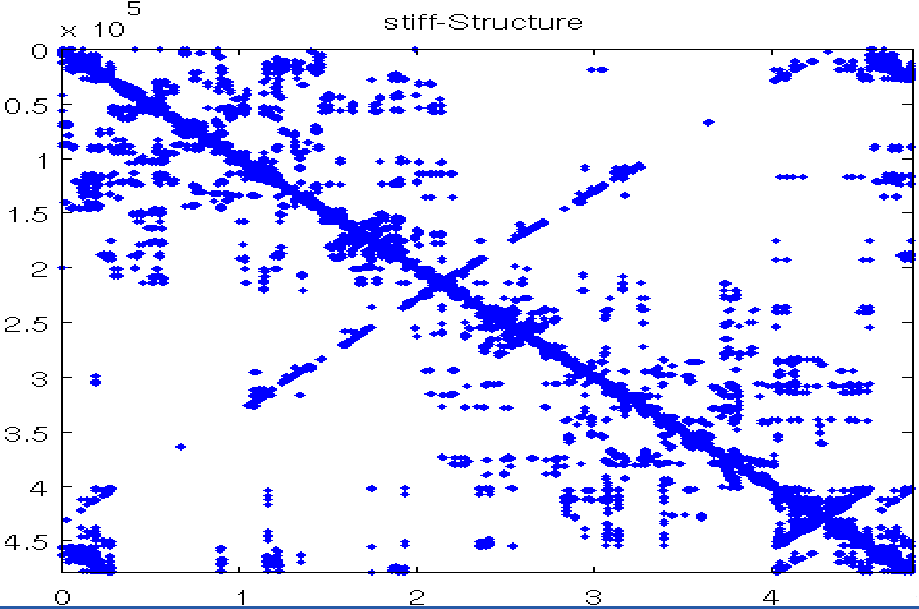
is complex symmetric of dimension up to $10,000,000$,

- M_s, M_f, K_f are real symm. pos. semidef. mass/stiffness matrices of structure and air, M_s is singular and diagonal, M_f is sparse. M_s is a factor $1000 - 10000$ larger than M_f.
- $K_s(\omega) = K_s(\omega)^T = K_1(\omega) + \imath K_2$.
- D_s is a real damping matrix, D_f is complex symmetric.
- D_{as} is real coupling matrix between structure and air.
- All or part of the matrices depend on geometry, topology and material parameters.

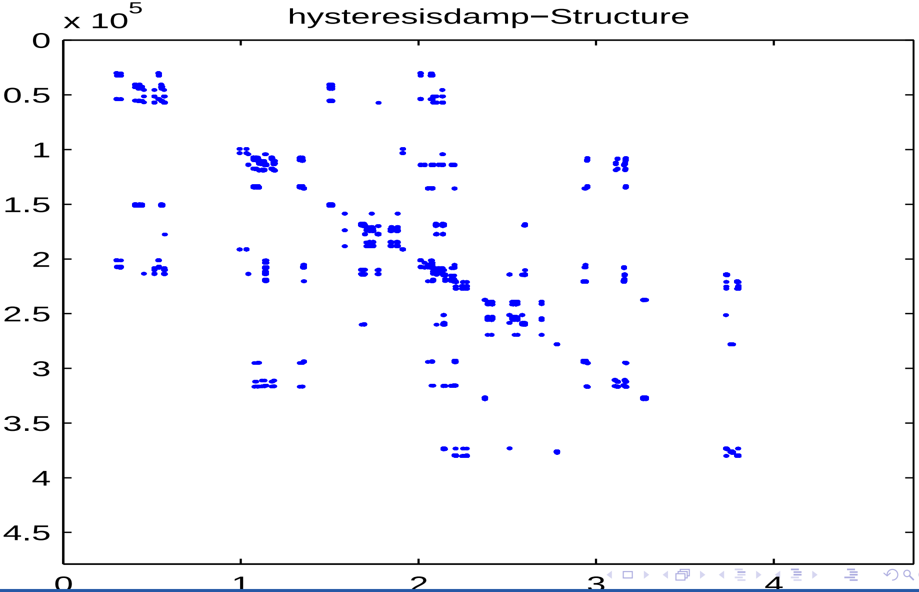
Sparsity of fluid mass matrix M_f



Sparsity of $K_1(\omega)$



Sparsity of K_2



Detailed tasks

- Solve for a given set of parameters α_i, $i = 1, 2, \dots,$ the linear system $P(\omega)u(\omega) = f(\omega)$, for $\omega = 0, \dots,\ 1000hz$ in small frequency steps.
- The parameters α_i are determined in a manual or (later) automatic optimization process, i.e. α_i and α_{i+1} are close.
- Parallelization in multi-processor multicore environment.
- Often many right hand sides (load vectors) $f(\omega)$.
- Accuracy goal: Relative residual 10^{-6}.

Difficulties and challenges

- Problems are badly scaled and get increasingly ill-conditioned when ω grows.
- For some parameter constellations the system becomes exactly singular with inconsistent right hand side.
- Direct solution methods would be ideal but work only work out-of-core.
- Small blocks of matrices are changed with α remaining system is the same.
- No multilevel or adaptive grid refinement available, methods must be purely matrix based.

Our contribution

- Generated and implemented subspace recycling Krylov subspace method with sparse out of core LDL^T preconditioner (MUMPS, PARDISO) for real part of linear system, i.e.

$$\tilde{P}(\omega) := -\omega^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \begin{bmatrix} K_1 & 0 \\ 0 & K_f \end{bmatrix}.$$

- For small ω only $2 - 4$ iteration steps per frequency are necessary.
- The number of iteration steps grows substantially for larger ω so that more and more new preconditioners are needed or the number of iterations or restarts increases.

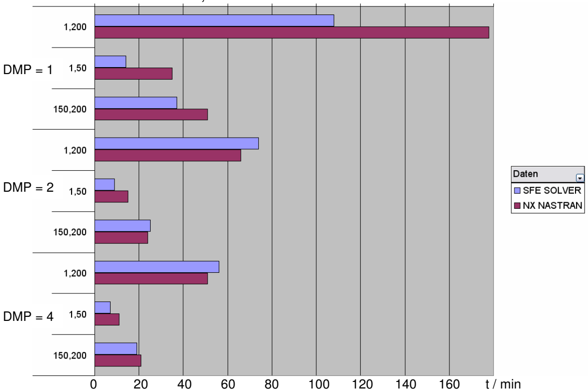
Zeitvergleich SOL 108

Rechner: AMD Optheron 2*DualCore 2600 Mhz

Betriebsystem: Suse Linux 10.0 Kernel 2.6

OpenCore : 1.5 GB , RAM: 8 GB

Modell: 219.432 DOFs, Load Cases: 1



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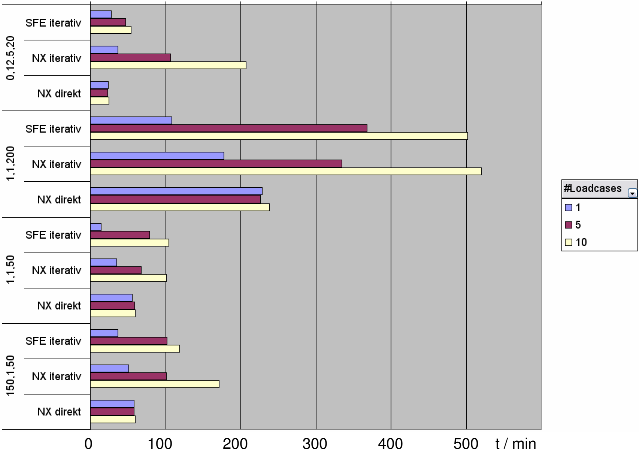
Comparison with NASTRAN

Number of processors: 1, DOFs: 219,432



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Consider nonlinear eigenvalue problem $P(\lambda)x = 0$, where the matrix polynomial

$$P(\lambda) := \lambda^2 \begin{bmatrix} M_s & 0 \\ 0 & M_f \end{bmatrix} + \lambda \begin{bmatrix} D_s & D_{as}^T \\ D_{as} & D_f \end{bmatrix} + \begin{bmatrix} K_s(\lambda) & 0 \\ 0 & K_f \end{bmatrix},$$

is **complex symmetric** and has dimension up to $10,000,000$, and all coefficients depend in part on α.

- Compute all smallest real eigenvalues in a given region of $\mathbb{C}$ and associated eigenvectors.
- Project the problem into the subspace spanned by these eigenvectors.
- Solve the second order differential algebraic system (DAE).
- Optimize the eigenfrequencies/acoustic field w.r.t. the set of parameters.

Methods directly for nonlinear problem (incomplete list). For surveys see M./Voss 2005 or Dissertation Schreiber 2008.

- Second order Arnoldi method Bai 2006
- Rational Krylov method Ruhe 1998, 2000
- Residual iteration method Neumaier 1985
- Newton-Type methods Schreiber/Schwetlick 2006, 2008,
- Rayleigh quotient iterations Schreiber 2008, Freitag/Spence 2007, 2008
- Jacobi-Davidson method Sleijpen/Van der Vorst et al 1996, Betcke/Voss 2004, Hochstenbach 2007
- Arnoldi type methods Voss 2003

Can we use these methods

- None of these methods can be applied directly.
- We need to improve convergence and preconditioning.
- We need better perturbation and error analysis.
- How can we guarantee a required accuracy.
- We need the methods in parallel on modern multi-processor, multi-core machines.

Challenges

- Guarantee that all desired eigenvalues are obtained.
- Guaranteed relative residual?
- Previously used decoupled methods for structure/fluid subsystems do not work appropriately.
- Problem is in some cases truely nonlinear since K_s may depend on λ.
- Eigenvalue is very ill-conditioned for some parameter sets.
- Mass matrix is block diagonal and singular. (Nullspace is available without extra computation.)
- Infinite eigenvalues have index 2.
- For some parameters α the whole matrix polynomial is singular.
- Locking and purging or deflation of converged eigenvalues?

Our contribution so far

- ▷ Analysis of singularity and structure.
- ▷ Trimmed structured linearization method to deal with singular mass matrix and singular pencil. Byers/M./Xu 2007
- ▷ Implicitly restarted Arnoldi for undamped system with guaranteed eigenvalues in a given interval for undamped systems. This is used as starting configuration in homotopy method for damped system. (Diploma thesis Elena Teidelt 2008)
- ▷ Newton-like methods and generalized Rayleigh quotient methods for general nonlinear systems (Dissertation Kathrin Schreiber May 2008)
- ▷ Special deflation methods for converged eigenvalues.

The classical companion linearization for polynomial eigenvalue problems

$$P(\lambda)x = \sum_{i=0}^{k} \lambda^i A_i x$$

is to introduce new variables

$$T = \left[\ y_1, y_2, \dots, y_k\ \right]^T = \left[\ x, \lambda x, \dots, \lambda^{k-1} x\ \right]^T$$

and to turn it into a generalized linear eigenvalue problem

$$L(\lambda)y := (\lambda \mathcal{E} + \mathcal{A})y = 0$$

of size $nk \times nk$.

Linearization

Definition: For a matrix polynomial $P(\lambda)$ of degree k, a matrix pencil $L(\lambda) = (\lambda\mathcal{E} + \mathcal{A})$ is called linearization of $P(\lambda)$, if there exist nonsingular unimodular matrices (i.e., of constant nonzero determinant) $S(\lambda), T(\lambda)$ such that

$$S(\lambda)L(\lambda)T(\lambda) = \text{diag}(P(\lambda), I_{(n-1)k}).$$

Properties of companion linearization

- Companion linearization preserves the algebraic and geometric multiplicities of all finite eigenvalues.
- There are some difficulties with multiple eigenvalues including ∞ and the singular part, Byers/M./Xu 2008.
- The geometric multiplicity of the eigenvalue ∞ and the sizes of singular blocks are not invariant under unimodular transformations.
- Companion linearization destroys the structure.

Example: Constrained Multi-body system

Consider the Euler-Lagrange equations of a linear,

$$\begin{aligned} \hat{M}\ddot{x} + \hat{D}\dot{x} + \hat{K}x + \hat{G}^T\mu &= f(t) \\ \hat{G}x &= g. \end{aligned}$$

The associated matrix polynomial is

$$P(\lambda) = \lambda^2 \begin{bmatrix} \hat{M} & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} \hat{D} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{K} & \hat{G}^T \\ \hat{G} & 0 \end{bmatrix}.$$

If $\hat{M}$ is positive definite and $\hat{G}$ has full row rank, then the companion form has a Kronecker block associated with ∞ of size 4.

Modified first order form

The first order formulation used in multibody dynamics only introduces $y = \dot{x}$ and not $\gamma = \dot{\mu}$.

$$
\begin{aligned}
M\dot{y} + D\dot{x} + Kx + G^T\mu &= f(t), \\
\dot{x} &= y, \\
Gx &= 0
\end{aligned}
$$

and the associated linear matrix pencil

$$
\tilde{L}(\lambda) = \lambda \begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} D & K & G^T \\ -I & 0 & 0 \\ 0 & G & 0 \end{bmatrix},
$$

has a Kronecker block at ∞ of size 3. Even smaller blocks can be achieved.

Example For the complex symmetric problem

$$(\lambda^2 M + \lambda D + K)x = 0$$

the companion linearizations

$$\lambda \begin{bmatrix} I & O \\ O & M \end{bmatrix} - \begin{bmatrix} O & I \\ K & -D \end{bmatrix}, \quad \lambda \begin{bmatrix} I & O \\ D & M \end{bmatrix} - \begin{bmatrix} O & I \\ K & O \end{bmatrix}$$

do not preserve the structure and the symmetric versions

$$\lambda \begin{bmatrix} K & O \\ O & M \end{bmatrix} - \begin{bmatrix} O & K \\ K & -D \end{bmatrix}, \quad \lambda \begin{bmatrix} M & O \\ D & M \end{bmatrix} - \begin{bmatrix} O & M \\ K & O \end{bmatrix}$$

may be singular. Linearization theory Mackey/Mackey/Mehl/M. 2006, Mackey/Higham/Tisseur 2006, Dopico/Mackey/Teran 2008 is needed.

Trimmed linearization

Consider the polynomial eigenvalue problem

$$(\sum_{i=0}^{k} A_i \lambda^i)x = 0.$$

- Can we do as in the multibody context?
- Can we remove singular and high index parts for the eigenvalue ∞ completely.
- In Byers/M./Xu 2008 a new trimmed linearization theory is developed.

A simple example

Consider the DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\mu} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Index reduction (inserting the derivatives of the second equation into the first) gives the first order DAE

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} f_1 - f_2 - \dot{f}_2 - \ddot{f}_2 \\ f_2 \end{bmatrix}.$$

This is first order, no first order formulation is necessary.

The associated matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 + \lambda + 1 & 1 \\ 1 & 0 \end{bmatrix}$$

has only the eigenvalue ∞. Using a unimodular transformation from the left with

$$Q(\lambda) = \begin{bmatrix} 1 & -(\lambda^2 + \lambda + 1) \\ 0 & 1 \end{bmatrix}$$

we obtain the first order

$$T(\lambda) = Q(\lambda)P(\lambda) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has only degree 0.
Is this a polynomial of degree 2, or 1 with leading coefficients 0.

Theorem Byers/M./Xu 2007

Let $A_i \in \mathbb{C}^{m,n}$ $i = 0, \ldots, k$. Then, the tuple $(A_k, \ldots, A_0)$ is unitarily equivalent to a matrix tuple $(\hat{A}_k, \ldots, \hat{A}_0) = (UA_kV, \ldots, UA_0V)$, where all terms $\hat{A}_i$, $i = 0, \ldots, k$, have the form

$$
\left[\begin{array}{cccc|c|cccc}
A & A & A & \ldots & \ldots & \ldots & A & A & A_l^{(i)} \\
A & A & A & \ldots & \ldots & \ldots & \iddots & A_{l-1}^{(i)} & 0 \\
A & A & A & \ldots & \ldots & \iddots & \iddots & 0 & 0 \\
\vdots & \vdots & \vdots & \iddots & \iddots & A_1^{(i)} & \iddots & \vdots & \vdots \\
\hline
\vdots & \vdots & \iddots & \iddots & A_0^{(i)} & 0 & \ldots & \vdots & \vdots \\
\hline
\vdots & \iddots & \iddots & \tilde{A}_1^{(i)} & \iddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots & \vdots & \vdots \\
A & \tilde{A}_{l-1}^{(i)} & 0 & \iddots & \iddots & \vdots & \vdots & \vdots & 0 \\
\tilde{A}_l^{(i)} & 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0
\end{array}\right],
$$

Properties of this staircase form

- Each of the blocks $A_j^{(i)}$ $i = 0, \ldots, k,\ j = 1, \ldots, l$ either has the form $\begin{bmatrix} \Sigma & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \end{bmatrix}$,
- Each of the blocks $\tilde{A}_j^{(i)}$ $i = 1, \ldots, k,\ j = 1, \ldots, l$ either has the form $\begin{bmatrix} \Sigma \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- For each j only one of the $A_j^{(i)}$ and $\tilde{A}_j^{(i)}$ is nonzero.
- In the tuple of middle blocks $(A_0^{(k)}, \ldots, A_0^{(k)})$ no k of the coefficients have a common nullspace.
- Is this all we need?

Deflation of singular parts

Let $P(\lambda)x(\lambda) \equiv 0$, $\tilde{x}(\lambda) := Vx(\lambda)$, where V is as in staircase form, and set

$$\begin{bmatrix} P_{11}(\lambda) & P_{12}(\lambda) & P_{13}(\lambda) \\ P_{21}(\lambda) & P_{22}(\lambda) & 0 \\ P_{31}(\lambda) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(\lambda) \\ x_2(\lambda) \\ x_3(\lambda) \end{bmatrix} = 0.$$

Then $x_1(\lambda) \equiv 0$, i.e. the right singular blocks of the polynomial $P(\lambda)$ are contained in the submatrix polynomial
$\begin{bmatrix} P_{12}(\lambda) & P_{13}(\lambda) \\ P_{22}(\lambda) & 0 \end{bmatrix}$.
All the eigenvalue information associated with finite nonzero eigenvalues is contained in $P_{22}(\lambda)$.

Properties of staircase form

- Staircase form allows to deflate long chains associated with ∞ and singular part.
- Concept can be extended to any other eigenvalue.
- Unfortunately for degree > 1 the information may not be complete, see following example.
- There is a good case, where all the information is available.

Bad Example

$$P(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

has double eigenvalues at $0, \infty$ with right and left chains

$$x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

associated with infinity and

$$z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

associated with 0. No two coefficients have a common nullspace.

We cannot reduce this matrix polynomial further with strong equivalence.

The complete information associated with $0, \infty$ is available if the staircase procedure ends up with a tuple of middle blocks $(A_0^{(k)}, \ldots, A_0^{(k)})$ which has a growing anti-triangular form

$$\left(\begin{bmatrix} \Sigma_k & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{11}^{(k-1)} & A_{12}^{(k-1)} & 0 & \cdots & 0 \\ A_{21}^{(k-1)} & \Sigma_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \ldots, \right.$$

$$\left. \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & \cdots & A_{1,k-1}^{(1)} & 0 \\ \vdots & \ddots & \cdots & A_{2,k-1}^{(1)} & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ A_{k-1,1}^{(1)} & A_{k-1,2}^{(1)} & \cdots & \Sigma_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} & \cdots & \cdots & A_{1,k}^{(0)} \\ \vdots & \ddots & \cdots & \cdots & A_{2,k}^{(0)} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ A_{k-1,1}^{(0)} & A_{k-1,2}^{(0)} & \cdots & A_{k-1,k-1}^{(0)} & A_{k-1,k}^{(0)} \\ A_{k,1}^{(0)} & A_{k,2}^{(0)} & \cdots & A_{k,k-1}^{(0)} & \Sigma_0 \end{bmatrix} \right)$$

Corollary If the middle block has growing anti-triangular form and is regular, then it only has simply eigenvalues associated with ∞.

Consider the associated eigenvalue problem $P_{22}(\lambda)\tilde{x} = 0$ with $\tilde{x} = [x_0^T, x_1^T, \ldots, x_k^T]^T$. Then we obtain a linear eigenvalue problem by introducing selected new variables (different from the usual companion form). Let

$$\begin{aligned} & z_{0,1} = \lambda x_0, \quad z_{0,2} = \lambda z_{0,1} = \lambda^2 x_0, \ldots, z_{0,k-1} = \lambda z_{0,k-2} = \lambda^{k-1} x_0, \\ & z_{1,1} = \lambda x_1, \quad z_{1,2} = \lambda z_{1,1} = \lambda^2 x_1, \ldots, z_{1,k-2} = \lambda z_{1,k-3} = \lambda^{k-2} x_1, \\ & \vdots \\ & z_{k-2,1} = \lambda x_{k-2}. \end{aligned}$$

Define

$$z = [x_0^T, x_1^T, \ldots, x_k^T, z_{0,1}^T, \ldots, z_{k-2,1}^T, z_{0,2}^T, \ldots, z_{k-3,2}^T, \ldots, z_{0,k-2}^T, z_{1,k-2}^T, z_{0,k-}$$

We call this trimmed linearization.

What do we learn from this?

- The trimmed linearization theory allows to remove singular parts and high index parts directly in the nonlinear system.
- In our application we can apply this technique directly to get regular structured linearizations.
- No null-space computation is necessary, since the kernel of M is available directly and exactly.
- Thus we can use structured methods for generalized eigenvalue problems.

Finding all evs in an interval/box

One of the goals is to find all eigenvalues in a real interval $[a, b]$ (undamped case) or a box of the complex plane.

- This is relatively easy for the undamped problem $\lambda^2 M - K$, we need to find all eigenvalues in a given real interval $[a, b]$.
- Carry out factorizations $P(a) = L(a)D(a)L(a)^T$ and $P(b) = L(b)D(b)L(b)^T$ and use inertia to determine number of eigenvalues in interval.
- Use several starts of implicitly restarted Arnoldi with shift-and-invert preconditioner for $P(a)^{-1}$, $P(b)^{-1}$, $P((b-a)/2)^{-1}$, $\ldots$ until all eigenvalues are found.
- In the general case we can use Bendixon's theorem or Gersgorin type results to analyse the number of eigenvalues.
- The computation can be done as in the interval case with several complex targets or using the sign function method.

Numerical results I, Dipl. E. Teidelt

- Balanced, scaled problem, infinite eigenvalues deflated.
- Matrix dimension: 478 788. Smallest 50 eigenvalues
- Condition number $kappa = ||x||^2/|lambda|^2 * |x' * M * x|$
- 1 factorization needed.

Ev. no	λ	residual	κ
1	$3.116828e+01$	$2.784910e-16$	$1.381088e+08$
2	$3.939059e+01$	$2.632970e-16$	$7.765305e+07$
3	$4.770588e+01$	$2.730574e-16$	$6.185278e+07$
⋮	⋮	⋮	⋮
6	$6.553705e+01$	$2.687169e-16$	$3.215041e+07$
7	$1.435197e+02$	$2.508269e-16$	$6.759916e+06$
⋮	⋮	⋮	⋮
48	$6.600993e+02$	$1.716196e-16$	$3.239416e+05$
49	$6.677409e+02$	$3.189563e-16$	$5.416045e+05$
50	$6.837248e+02$	$2.204968e-13$	$4.647000e+05$

Numerical results II Dipl. E. Teidelt

- Balanced, scaled problem, infinite eigenvalue deflated.
- Matrix dimension: 478 788
- All eigenvalues in $[400, 650]$
- 31 eigenvalues found, all converged.
- 2 factorizations needed.

Ev. no	λ	residual	κ
1	$4.007569e+02$	$5.566134e-16$	$1.747769e+06$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
30	$6.427129e+02$	$1.286406e-08$	$2.231207e+05$
31	$6.431337e+02$	$1.149491e-08$	$2.423338e+05$

General quadratic problem, in progress

- Bring in damping via homotopy. Solve

$$(1 - t_i)(\lambda^2 M + K) + t_i(\lambda D), \;\; t_0, \ldots, t_\ell \in [0, 1]$$

- Use computed symmetric factorizations as long as possible.
- Use new symmetric factorizations of real part.
- Recycle Krylov subspaces when possible.
- Follow eigenvalue curves with stepsize control to guarantee that no eigenvalue is missed.
- Use Newton method for fully nonlinear problem.

Nonlinear Newton evp. solver.

Truely nonlinear evp $T(\lambda)x = (\lambda^2 M + \lambda D + K(\lambda))x = 0$.
Apply Newton to function

$$f_w(x, \lambda) = \begin{bmatrix} T(\lambda)x \\ w^H x - 1 \end{bmatrix} = 0.$$

The Newton system for $\lambda_{k+1} = \lambda_k + \mu_k$ and $x_{k+1} = x_k + s_k$ is

$$\begin{bmatrix} T(\lambda_k) & \dot{T}(\lambda_k)x_k \\ w^H & 0 \end{bmatrix} \begin{bmatrix} s_k \\ \mu_k \end{bmatrix} = - \begin{bmatrix} T(\lambda_k)x_k \\ w^H x_k - 1 \end{bmatrix}$$

or

$$\begin{aligned} \lambda_{k+1} &= \lambda_k - \frac{1}{w^H T(\lambda_k)^{-1} \dot{T}(\lambda_k) x_k} \\ x_{k+1} &= (\lambda_k - \lambda_{k+1}) T(\lambda_k)^{-1} \dot{T}(\lambda_k) x_k. \end{aligned}$$

Dissertation Schreiber 2008

- Conditions for local quadratic convergence of Newton.
- Proof of local cubic connvergence of two-sided nonlinear Jacobi-Davidson and Rayleigh quotient interation.
- Implementation of method.
- Comparison of methods Newton type, Jacobi Davidson, nonlinear two-sided Rayleigh quotients.
- Special methods for complex symmetric problems.

Difficulties

- Small homotopy steps necessary to track eigenvalues of polynomial and nonlinear problem.
- Need to store intermediate Krylov subspaces to make efficient restart.
- Need to use out-of-core sparse solvers.
- Need to get into convergence intervals for Newton, JD.
- No global analysis and industrial implementation yet.

Model/modal reduction

- After desired eigenvalues and corresponding deflating subspaces $U = [u_1, \ldots, k]$ have been computed, the projected coupled DAE system

$$U^T M(\alpha) U \ddot{z} + U^T D(\alpha) U \dot{z} + U^T K(\alpha) U z = U^T f$$

  is formed and eigenvalue/frequency optimization is done on this system.
- The decoupled projection does not work.
- We would really need nonlinear model reduction.
- We need to use the fact that only a small part of the system is changed in every optimization step.
- We need to integrate ev computation, gradient computation, discretization.
- A multilevel approach would be great.

Conclusions

- Industrial applications lead to nice mathematical questions.
- Commercially available codes are not satisfactory.
- Coupled nonlinear eigenvalue with a structured part.
- Structure preserving linearization techniques have been derived for polynomial part, but infinite eigenvalues and singularities need to be deflated first.
- Homotopy and Newton like method are developed.
- Industrial production code development is a challenge.

Acry for help.

- Really nonlinear eigenvalue solvers are not well understood.
- Conditioning and accuracy of eigenvalues is not well understood.
- Jacobi-Davidson method need to be improved to be compatible.
- Deflation of converged eigenvalues need to be improved.
- Subspace recycling in homotopy, Newton, and optimization methods needs to be improved.
- Linear system and eigenvalue solvers need to be better adapted.
- Adaptive methods for PDE eigenvalue problems are needed.

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Thank you very much
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