Quadratic two-parameter eigenvalue problem

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Outline

1. Quadratic two-parameter eigenvalue problem
2. Properties of linearization for QMEP
3. Algorithm for the extraction of the common regular part

Quadratic two-parameter eigenvalue problem

- Quadratic two-parameter eigenvalue problem:

$$\overbrace{(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)}^{W_1(\lambda,\mu)}\, x = 0$$
$$\overbrace{(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)}^{W_2(\lambda,\mu)}\, y = 0 \qquad \text{(QMEP)}$$

  where $A_i, B_i, C_i, D_i, E_i, F_i$ are $n_i \times n_i$ matrices, $\lambda, \mu \in \mathbb{C}$, $x \in \mathbb{C}^{n_1}$, $y \in \mathbb{C}^{n_2}$.
- Eigenvalue: a pair (λ, μ) which satisfies (QMEP) for nonzero x and y.
- Equivalent problem in generic case:
  finding common zeros of polynomials $p_1(\lambda, \mu) = \det(W_1(\lambda, \mu))$ and $p_2(\lambda, \mu) = \det(W_2(\lambda, \mu))$.
- number of eigenvalues in generic case is $4n_1n_2$
- **Goal: compute all eigenvalues** (λ, μ)

Linearization for QMEP

- Possible linearization for QMEP:

$$\left(\overbrace{\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}^{A^{(1)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_1 & \frac{1}{2}E_1 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(1)}} + \mu \overbrace{\begin{bmatrix} 0 & \frac{1}{2}E_1 & F_1 \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix}}^{C^{(1)}}\right) w_1 = 0$$

$$\left(\overbrace{\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}^{A^{(2)}} + \lambda \overbrace{\begin{bmatrix} 0 & D_2 & \frac{1}{2}E_2 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{B^{(2)}} + \mu \overbrace{\begin{bmatrix} 0 & \frac{1}{2}E_2 & F_2 \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix}}^{C^{(2)}}\right) w_2 = 0,$$

  blocks in row i are $n_i \times n_i$ complex matrices. Linearizations are of the dimension $3n_i \times 3n_i$.
- this is standard two-parameter eigenvalue problem
- problem is singular

Two parameter eigenvalue problem

- Two parameter eigenvalue problem:

$$\left(A^{(1)} + \lambda B^{(1)} + \mu C^{(1)}\right) w_1 = 0$$
$$\left(A^{(2)} + \lambda B^{(2)} + \mu C^{(2)}\right) w_2 = 0 \qquad \text{(MEP)}$$

- Eigenvalue: a pair (λ, μ) which satisfies (MEP) for nonzero w_1 and w_2.
- Eigenvector: the tensor product $w_1 \otimes w_2$.

Tensor product approach

- We define Δ_i matrices on the space $\mathbb{C}^{3n_1 \times 3n_2}$

$$\begin{aligned}
\Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)} \\
\Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)} \\
\Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}.
\end{aligned}$$

- MEP is nonsingular $\Longleftrightarrow$ some combination of Δ_i (usually Δ_0) is nonsingular.
- MEP is eiquivalent to coupled GEP

$$\begin{aligned}
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z,
\end{aligned} \qquad (\Delta)$$

where $z = w_1 \otimes w_2$.
- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute.

Singular two-parameter eigenvalue problem

- every combination of Δ_i is singular
- pencils $\lambda\Delta_0 - \Delta_1$ and $\mu\Delta_0 - \Delta_2$ are singular
- eigenvalue ω is a finite regular eigenvalue of matrix pencil $\lambda B - A$ if and only if

$$\operatorname{rank}(\omega B - A) < \max_{s\in\mathbb{C}} \operatorname{rank}(sB - A) = n_r.$$

- Model updating (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.
- Spectrum of delay-differential equations (Jahrlebring 2008)
- QMEP

Finite regular eigenvalue of two pencils

Definition

Pair (λ, μ) is a finite regular eigenvalue of pencils $\lambda\Delta_0 - \Delta_1$ and $\mu\Delta_0 - \Delta_2$, if and only if

1. λ is a finite regular eigenvalue of $\lambda\Delta_0 - \Delta_1$,
2. μ is a finite regular eigenvalue of $\mu\Delta_0 - \Delta_2$,
3. there exists common proper eigenvector z in the intersection of regular parts of pencils $\lambda\Delta_0 - \Delta_1$, $\mu\Delta_0 - \Delta_2$ for which

$$\begin{aligned}(\lambda\Delta_0 - \Delta_1)z &= 0,\\ (\mu\Delta_0 - \Delta_2)z &= 0.\end{aligned}$$

Kronecker canonical form

Definition

Let $\lambda B - A \in \mathbb{C}^{m\times n}$ be a matrix pencil. There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$, such that

$$P^{-1}(B - \lambda A)Q = \widetilde{B} - \lambda\widetilde{A} = \mathrm{diag}(B_1 - \lambda A_1, \ldots, B_b - \lambda A_b)$$

is the Kronecker canonical form, where $B_i - \lambda A_i$ is one of regular blocks

$$J_j(\alpha) = \begin{bmatrix} \alpha - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha - \lambda \end{bmatrix}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix},$$

or one of singular blocks

$$L_j = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad L_j^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix}.$$

Kronecker canonical structure for QMEP

Studying block structure of matrices Δ_0, Δ_1 and Δ_2 we prove the following theorem.

Theorem

Kronecker canonical form of pencils $\lambda\Delta_0 - \Delta_1$ and $\mu\Delta_0 - \Delta_2$ has n_1n_2 L_0, n_1n_2 L_0^T, $2n_1n_2$ N_2 blocks and the finite regular part of size $4n_1n_2$in generic case.

- Δ_0 is of rank $6n_1n_2$, Δ_1 and Δ_2 are of rank $8n_1n_2$
- common kernel of Δ_0 and Δ_1 is of the dimension n_1n_2
- common kernel of Δ_0^T and Δ_1^T is of the dimension n_1n_2
- pencil $\lambda\Delta_0^T - \Delta_1^T$ has at least $2n_1n_2$ first root vectors for the eigenvalue ∞

Eigenvalues for QMEP

Theorem

The eigenvalues of QMEP are common regular eigenvalues of matrix pencils $\lambda\Delta_0 - \Delta_1$ and $\lambda\Delta_0 - \Delta_2$.

- Eigenvector of the form

$$z = \begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix} \otimes \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix}$$

  is the eigenvector for (λ, μ), which we get from linearization.
- Vector z has nonzero first block component.
- Vectors in the kernels of Δ_1 and Δ_2 have zero first block component.
- Rank decreases: $\operatorname{rank}(\lambda\Delta_0 - \Delta_1) < 8n_1n_2$, $\operatorname{rank}(\mu\Delta_0 - \Delta_2) < 8n_1n_2$.

QMEP is singular problem

Every combination of matrices Δ_i is such that

$$(\alpha_0\Delta_0^T+\alpha_1\Delta_1^T+\alpha_2\Delta_2^T)\left(a\begin{bmatrix}0\\x\\x\end{bmatrix}\otimes\begin{bmatrix}0\\y\\y\end{bmatrix}+b\begin{bmatrix}0\\x\\0\end{bmatrix}\otimes\begin{bmatrix}0\\y\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\x\end{bmatrix}\otimes\begin{bmatrix}0\\0\\y\end{bmatrix}\right)=0,$$

where $a=\alpha_1\alpha_2$, $b=\alpha_1^2-\alpha_1\alpha_2$, and $c=\alpha_2^2-\alpha_1\alpha_2$.
Our problem is therefore singular.

Kronecker canonical like form

Definition

Possible Kronecker canonical like form for the matrix pencil $\lambda B - A$ is the following

$$P^*(\lambda B - A)Q = \left[\begin{array}{cc|cc} \lambda B_\mu - A_\mu & & & \\ \times & \lambda B_\infty - A_\infty & & \\ \hline \times & \times & \lambda B_f - A_f & \\ \times & \times & \times & \lambda B_\epsilon - A_\epsilon \end{array}\right].$$

Pencils $\lambda B_\mu - A_\mu$, $\lambda B_\infty - A_\infty$, $\lambda B_f - A_f$, and $\lambda B_\epsilon - A_\epsilon$ contain the left singular structure, the infinite regular structure, the finite regular structure, and the right singular structure, respectively. Matrices P and Q are orthogonal.

We are interested in finite regular structure contained in lower block together with right singular part.

Row collumn compression

$D_0 = \Delta_0$, $D_1 = \Delta_1$

Repeat,

1.
   1. Matrix D_0 has size $m \times n$ and row rank r.
   2. If matrix D_0 has full row rank, exit and return D_0, D_1.
2. Compute row compression of matrix D_0, $U_0^* D_0 = \begin{matrix} r \\ m-r \end{matrix} \overset{n}{\begin{bmatrix} \times \\ 0 \end{bmatrix}}$.

   Compute block H of $U_0^* D_1 = \begin{matrix} r \\ m-r \end{matrix} \overset{n}{\begin{bmatrix} \times \\ H \end{bmatrix}}$ and compress it to full column rank c with V_1.

   We get $U_0^*(\lambda D_0 - D_1)V_1 = \begin{matrix} r \\ m-r \end{matrix} \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & \widehat{D}_0 \\ 0 & 0 \end{bmatrix}} - \begin{matrix} r \\ m-r \end{matrix} \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & \widehat{D}_1 \\ \times & 0 \end{bmatrix}}$.
3. Assign $D_0 = \widehat{D}_0$, $D_1 = \widehat{D}_1$ and proceed to 1.

Collumn row compression

$D_0 = \Delta_0$, $D_1 = \Delta_1$
Repeat,

1.
   1. Matrix D_0 has size $m \times n$ and column rank r.
   2. If matrix D_0 has full column rank, exit and return D_0, D_1.
2. Compute column compression of matrix D_0,

   $D_0 = D_0 V_0 = m \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & 0 \end{bmatrix}}$. Compute block H of

   $D_1 V_0 = m \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & H \end{bmatrix}}$ and compress it to the full row rank r with U_1.

   We get $U_1^*(\lambda D_0 - D_1)V_0 = \begin{matrix} r \\ m-r \end{matrix} \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & 0 \\ \widehat{D}_0 & 0 \end{bmatrix}} - \begin{matrix} r \\ m-r \end{matrix} \overset{\begin{matrix} c & n-c \end{matrix}}{\begin{bmatrix} \times & \times \\ \widehat{D}_1 & 0 \end{bmatrix}}$.
3. Assign $D_0 = \widehat{D}_0$, $D_1 = \widehat{D}_1$ and proceed to 1.

Algorithm for the extraction of the common regular part

$P = I_m$, $Q = I_n$, Δ_0 is of the size $m \times n$

1. Separate infinite and finite part.
   (a) Apply algorithm Row collumn compression on $\lambda P^* \Delta_0 Q - P^* \Delta_1 Q$ and $\mu P^* \Delta_0 Q - P^* \Delta_2 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Join the spaces.
   Compute orthogonal matrix Q such that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and orthogonal matrix P such that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.
   (c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and proceed to 2. Otherwise proceed to (a).
2. Separate the finite regular part from the right singular part.
   (a) Apply algorithm Collumn row compression on $\lambda P^* \Delta_0 Q - P^* \Delta_1 Q$ and $\mu P^* \Delta_0 Q - P^* \Delta_2 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Join the spaces.
   Compute orthogonal matrix Q such that $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and orthogonal matrix P such that $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$.
   (c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and exit. Otherwise proceed to (a).

Hypotesis

Algorithm returns $\widetilde{\Delta_0} = P^*\Delta_0 Q$, $\widetilde{\Delta_1} = P^*\Delta_1 Q$, $\widetilde{\Delta_2} = P^*\Delta_2 Q$, such that $\widetilde{\Delta_0}^{-1}\widetilde{\Delta_1}$ and $\widetilde{\Delta_0}^{-1}\widetilde{\Delta_2}$ commute.

- Hypotesis holds for QMEP in generic case.

$$\widetilde{\Delta_0}^{-1}\widetilde{\Delta_1}\widetilde{\Delta_0}^{-1}\widetilde{\Delta_2}z = \lambda\mu z = \widetilde{\Delta_0}^{-1}\widetilde{\Delta_2}\widetilde{\Delta_0}^{-1}\widetilde{\Delta_1}z$$

- We can solve coupled GEP in a standard way.

Model updating

Some results about special symmetric singular problems can be found in (Cottin 2001).

- All Δ_i matrices are symmetric and $\mathrm{Im}(\Delta_1), \mathrm{Im}(\Delta_2) \subseteq \mathrm{Im}(\Delta_0)$.
- One can use a generalised inverse of Δ_0 to obtain matrices $\Delta_0^+\Delta_0$, $\Delta_0^+\Delta_1$, and $\Delta_0^+\Delta_1$.
- Matrices are of the form

$$\begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \end{array},$$

  where k is the dimension of $\ker \Delta_0$.
- We continue with $m \times m$ submatrices $\widehat{\Delta_0} = I_m$, $\widehat{\Delta_1}$, and $\widehat{\Delta_2}$.
- When all eigenvalues are semisimple, matrices $\widehat{\Delta_1}$ and $\widehat{\Delta_2}$ commute. This is only a special case of our algorithm for extraction of common regular part.

Conclusions

- Solution for QMEP in the generic case.
- We proposed an algorithm for solving SMEP.
- We are able to prove that our algorithm works in some special cases.

Work in progress: SMEP

- Regular eigenvalues for SMEP?
- How to do extraction algorithm simultaneously?
- Prove that our algorithm works in general.
- $\vdots$

Numerical example

- $$\left(\begin{bmatrix}-3 & \\ & -1\end{bmatrix} + \lambda\begin{bmatrix}7 & \\ & 1\end{bmatrix} + \mu\begin{bmatrix}4 & \\ & 4\end{bmatrix} + \lambda^2\begin{bmatrix}6 & \\ & 2\end{bmatrix} + \lambda\mu\begin{bmatrix}10 & \\ & 1\end{bmatrix} + \mu^2\begin{bmatrix}4 & \\ & -3\end{bmatrix}\right) x = 0,$$
  $$\left(\begin{bmatrix}-1 & \\ & -1\end{bmatrix} + \lambda\begin{bmatrix}-1 & \\ & 2\end{bmatrix} + \mu\begin{bmatrix}2 & \\ & -1\end{bmatrix} + \lambda^2\begin{bmatrix}2 & \\ & 3\end{bmatrix} + \lambda\mu\begin{bmatrix}7 & \\ & 7\end{bmatrix} + \mu^2\begin{bmatrix}3 & \\ & 2\end{bmatrix}\right) y = 0.$$
  We multiply matrices in both equation with arbitrary orthogonal matrices and get a problem with known solutions.
- Matrices Δ_0, Δ_1, Δ_2 obtained from linearization are of the size 36×36.
- Using our algorithm we obtain matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ of the size 16×16.
- Matrices $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute.
- We get exactly 16 common regular eigenvalues.

The discrete spectrum of two-parameter linear polynomial

For Further Reading

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