A quadratic multiparameter eigenvalue problem arising in DDEs

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IWASEP 7
June 11, 2008

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Overview

- Critical delays of DDEs lead to QMEP
- Existing methods for DDEs critical delays can be understood in QMEP terms
- Numerical solution method for QMEP
- Generalization: PMEP

Critical delays in DDEs

Consider delay differential equation (DDE)

$$A\dot{x}(t) = Bx(t) + Cx(t - \tau)$$

$A,\ B,\ C \in \mathbb{R}^{n \times n}$, delay $\tau \geq 0$

Characteristic equation $(x(t) = e^{\lambda\tau}x)$

$$\lambda Ax = Bx + e^{-\tau\lambda}Cx$$

In particular interested in critical delays:
τ such that there is purely imaginary $\lambda = i\omega$

Methods for critical delays in DDEs

Louisell (2001): Imaginary eigenvalues of $\lambda x(t) = Ax(t) + e^{-\tau\lambda}Bx$ are eigenvalues of

$$\begin{bmatrix} A \otimes I & B \otimes I \\ -I \otimes B & -I \otimes A \end{bmatrix}$$

Chen, Gu, Nett (1995): if $e^{-\tau\lambda}$ of form $e^{-i\omega}$ then in spectrum of pair

$$\begin{bmatrix} I & 0 \\ 0 & B \otimes I \end{bmatrix}, \qquad \begin{bmatrix} 0 & I \\ -I \otimes B^T & -A \otimes I - I \otimes A^T \end{bmatrix}$$

Questions:

- Is there a common framework for this?
- Computational approaches?

Critical delays in DDEs

Delay eigenvalue problem

$$\lambda Ax = Bx + e^{-\tau\lambda}Cx$$

Critical delays: τ such that there is purely imaginary $\lambda = i\omega$

Denote $\mu = e^{-\tau\lambda}, \quad y = \overline{x}$

Equation and complex conjugate:

$$\begin{aligned} \lambda Ax &= Bx + \mu Cx \\ -\lambda Ay &= By + \mu^{-1}Cy \end{aligned}$$

Critical delays in DDEs

$\lambda A x = B x + e^{-\tau\lambda} C x$ leads to

$$\begin{aligned} \lambda A x &= B x + \mu C x \\ -\lambda A y &= B y + \mu^{-1} C y \end{aligned}$$

For critical delays:

1. solve for λ, select imaginary ones
2. for every such λ, determine τ such that for $\mu = e^{-\tau\lambda}$:
   $\lambda A - B - \mu C$ and $\lambda A + B + \mu^{-1} C$ singular
3. x corresponding eigenvector

QMEP

$$\begin{aligned} \lambda A x &= Bx + \mu C x \\ -\lambda A y &= By + \mu^{-1} C y \end{aligned}$$

Quadratic two-parameter eigenproblem (QMEP):

$$\begin{aligned} (B - \lambda A x - \mu C)\, x &= 0 \\ (C + \mu B + \lambda \mu A)\, y &= 0 \end{aligned}$$

Linear two-parameter eigenvalue problem (MEP):

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$$

General QMEP:

$$\begin{aligned} p(\lambda, \mu) x := (A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda \mu E_1 + \mu^2 F_1)\, x &= 0 \\ q(\lambda, \mu) y := (A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda \mu E_2 + \mu^2 F_2)\, y &= 0 \end{aligned}$$

MEP

Linear two-parameter multiparameter eigenvalue problem (MEP):

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x \\ A_2 y &= \lambda B_2 y + \mu C_2 y \end{aligned}$$

Matrix determinants:

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \end{aligned}$$

$\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute; associated 2 GEPs: Atkinson (1972)

$$\begin{aligned} \Delta_1(x \otimes y) &= \lambda\, \Delta_0(x \otimes y) \\ \Delta_2(x \otimes y) &= \mu\, \Delta_0(x \otimes y) \end{aligned}$$

Simultaneous diagonalization

Bunse-Gerstner, Byers, Mehrmann (1993), H., Košir, Plestenjak (2005)

Numerical solution for QMEP

Subspace method:

- subspace expansion: given $(\theta, \eta, u, v) \approx (\lambda, \mu, x, y)$, enlarge subspaces $\mathcal{U}$, $\mathcal{V}$
  $\Rightarrow$ Jacobi–Davidson type expansion
- subspace extraction: given $\mathcal{U}$, $\mathcal{V}$, select $(\theta, \eta, u, v) \approx (\lambda, \mu, x, y)$ with $u \in \mathcal{U}$ and $v \in \mathcal{V}$
  $\Rightarrow$ standard, harmonic, refined extraction

Jacobi–Davidson for QMEP

Suppose have approximation $(\theta, \eta, u, v) \approx (\lambda, \mu, x, y)$

Want $s \perp u, \;\; t \perp v$:

$$\begin{aligned} p(\lambda,\mu)(u+s) &= 0 \\ q(\lambda,\mu)(v+t) &= 0 \end{aligned}$$

rewrite:

$$\begin{aligned} p(\theta,\eta)s &= -p(\theta,\eta)u - (\lambda-\theta)p_\lambda(\theta,\eta)u - (\mu-\eta)p_\mu(\theta,\eta)u + \text{h.o.t.} \\ p(\theta,\eta)t &= -q(\theta,\eta)v - (\lambda-\theta)q_\lambda(\theta,\eta)v - (\mu-\eta)q_\mu(\theta,\eta)v + \text{h.o.t.} \end{aligned}$$

correction equation:

$$P \begin{bmatrix} p(\theta,\eta) & \\ & q(\theta,\eta) \end{bmatrix} P \begin{bmatrix} s \\ t \end{bmatrix} = - \begin{bmatrix} p(\theta,\eta)u \\ q(\theta,\eta)v \end{bmatrix}$$

Co-dim-2 projector $P = I - W(Z^*W)Z^*$

$$W = \text{span}\left(\begin{bmatrix} p_\lambda(\theta,\eta)u \\ q_\lambda(\theta,\eta)v \end{bmatrix} \; \begin{bmatrix} p_\mu(\theta,\eta)u \\ q_\mu(\theta,\eta)v \end{bmatrix} \right), \quad Z = \left(\begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} \right)$$

Standard extraction for QMEP

Determine $\theta, \ \ \eta, \ \ u \in \mathcal{U}, \ \ v \in \mathcal{V}$ such that

$$p(\theta,\eta)u \perp \mathcal{U} \qquad q(\theta,\eta)v \perp \mathcal{V}$$

With $u = Uc, \ \ V = Vd \Rightarrow$ small, projected QMEP:

$$\begin{aligned} U^* p(\theta,\eta) Uc &= 0 \\ V^* q(\theta,\eta) Vd &= 0 \end{aligned}$$

Alternatives to standard extraction

As for standard, generalized eigenvalue problem:

- standard extraction generally favorable for exterior eigenvalues
- not for interior eigenvalues
- reason: no bound on residual norms

  $$\|p(\sigma,\tau)u\|, \qquad \|q(\sigma,\tau)v\|$$

- instead: (1) refined extraction
- instead: (2) harmonic extraction

Refined extraction for QMEP

$$\min_{u \in \mathcal{U}} \|p(\sigma, \tau)u\| \qquad \min_{v \in \mathcal{V}} \|q(\sigma, \tau)v\|$$

+ efficient implementation with QR-decompositions
+ small residual norms
− true eigenvectors don't minimize $\Rightarrow$ update target (σ, τ)

Elegant approach: harmonic extraction

Harmonic extraction for GEP $Ax = \lambda Bx$

Shift-and-invert property:

$$(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x$$

Galerkin condition:

$$(A - \tau B)^{-1} Bu - (\theta - \tau)^{-1} u \perp \mathcal{V}$$

To avoid inversion:

$$(A - \tau B)^{-1} Bu - (\theta - \tau)^{-1} u \perp (A - \tau B)^{*}(A - \tau B)\mathcal{U}$$

equivalent with

$$(A - \theta B)u \perp (A - \tau B)\mathcal{U}$$

Generalization of harmonic extraction for MEP

For $Ax = \lambda Bx + \mu Cx$, no shift-and-invert property of type $(A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x$ seems feasible

Instead generalize $(A - \theta B) u \perp (A - \tau B)\mathcal{U}$:

$$\begin{aligned} (A_1 - \theta B_1 - \eta C_1) u &\perp (A_1 - \sigma B_1 - \tau C_1)\mathcal{U} \\ (A_2 - \theta B_2 - \eta C_2) v &\perp (A_2 - \sigma B_2 - \tau C_2)\mathcal{V} \end{aligned}$$

Justification:

$$(A_1 - \sigma B_1 - \tau C_1) u + (\theta - \sigma) B_1 u + (\eta - \tau) C_1 u \perp \mathcal{Q}$$

with $\mathcal{Q} := \mathrm{span}(A_1 - \sigma B_1 - \tau C_1)\mathcal{U}$, implies

$$\begin{aligned} \|(A_1 - \sigma B_1 - \tau C_1) u\| &\le |\theta - \sigma|\, \|Q^* B_1 u\| + |\eta - \tau|\, \|Q^* C_1 u\| \\ &\le |\theta - \sigma|\, \|Q^* B_1 U\| + |\eta - \tau|\, \|Q^* C_1 U\| \end{aligned}$$

Efficient implementation with QR-decompositions

H., Plestenjak (2008)

Harmonic extraction for QMEP

$$\begin{aligned} p(\theta,\eta)u &\perp p(\sigma,\tau)\,\mathcal{U} \\ q(\theta,\eta)v &\perp q(\sigma,\tau)\,\mathcal{V} \end{aligned}$$

Justification: up to h.o.t.:

$$p(\sigma,\tau)u + (\theta-\sigma)p_\lambda u + (\eta-\tau)p_\mu u \perp \mathcal{Q} := \mathrm{span}(p(\sigma,\tau)\mathcal{U})$$

implies

$$\begin{aligned} \|p(\sigma,\tau)u\| &\lesssim |\theta-\sigma|\,\|Q^* p_\lambda(\sigma,\tau)u\| + |\eta-\tau|\,\|Q^* p_\mu(\sigma,\tau)u\| \\ &\lesssim |\theta-\sigma|\,\|Q^* p_\lambda(\sigma,\tau)U\| + |\eta-\tau|\,\|Q^* p_\mu(\sigma,\tau)U\| \end{aligned}$$

similar bound for $\|q(\sigma,\tau)v\|$

Solution of small (projected) QMEP

But how to solve small QMEP in:

- standard extraction:

$$\begin{aligned} U^* p(\theta,\eta) U c &= 0 \\ V^* q(\theta,\eta) V d &= 0 \end{aligned}$$

- harmonic extraction:

$$\begin{aligned} Q_1^* p(\theta,\eta) U c &= 0 \\ Q_2^* q(\theta,\eta) V d &= 0 \end{aligned}$$

Solution of "DDE special case" of QMEP

Our special case:

$$\begin{array}{lllll}(A_1 & +\lambda B_1 x & +\mu C_1 & &)\,x = 0 \\ (A_2 & & +\mu C_2 & +\lambda\mu E_2 &)\,y = 0\end{array}$$

Slightly more general:

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1 + \lambda\mu E_1)\,x &= 0 \\ (A_2 + \lambda B_2 + \mu C_2 + \lambda\mu E_2)\,y &= 0\end{aligned}$$

Lemma (gives 2 QEPs in 1 parameter)

(a) $[\lambda^2(E_1\otimes B_2 - B_1\otimes E_2) + \lambda(E_1\otimes A_2 - A_1\otimes E_2 + C_1\otimes B_2 - B_1\otimes C_2)$
$+(C_1\otimes A_2 - A_1\otimes C_2)]\,(x\otimes y) = 0$

(b) $[\mu^2(E_1\otimes C_2 - C_1\otimes E_2) + \mu(E_1\otimes A_2 - A_1\otimes E_2 + B_1\otimes C_2 - C_1\otimes B_2)$
$+(B_1\otimes A_2 - A_1\otimes B_2)]\,(x\otimes y) = 0$

Generalization: PMEP

So far:

$$A\dot{x}(t) = Bx(t) + Cx(t-\tau)$$

Neutral DDEs with commensurate delays

$$\sum_{k=0}^{m} B_k \dot{x}(t-k\tau) = \sum_{k=0}^{m} A_k x(t-k\tau)$$

Characteristic equation

$$\left(A_0 x + \sum_{k=1}^{m} \mu^k A_k - \sum_{k=0}^{m} \lambda \mu^k B_k \right) x = 0$$

PMEP: polynomial multiparameter eigenvalue problem

Experiment delay case QMEP: dense

$$\begin{array}{llllll} (A_1 & +\lambda B_1 x & +\mu C_1 & &)\,x = 0 \\ (A_2 & & +\mu C_2 & +\lambda\mu E_2 &)\,y = 0 \end{array}$$

Work:

- Kronecker products size $n_1 n_2$
- linearization (QEP $\rightarrow$ EP) size $2n_1 n_2$
- work "$\mathcal{O}(8 n_1^3 n_2^3)$"

$n_1 = n_2$	time (s)
5	0.026
10	0.73
20	91.5

$\Rightarrow$ in subspace method, have to keep size of subspaces modest

Conclusions

- Critical delays in DDEs give rise to QMEP
- Seemingly different approaches for critical delays: all connected via QMEP framework
- numerical solution method for QMEP
- generalizations to PMEP

Some references

DDEs:
Chen, Gu, Nett (1995)
Niculescu (1998)
Louisell (2001)
Gu, Kharitonov, Chen (2003)
Fu, Niculescu, Chen (2006)
Jarlebring (2006, 2007, 2008)
Michiels, Niculescu (2007)
Ergenc, Olgac, Fazelina (2007)

MEP:
Atkinson (1972)
Bunse-Gerstner, Byers, Mehrmann (1993)
H., Košir, Plestenjak (2005); H., Plestenjak (2008)

QMEP:

This talk: work with Jarlebring, Muhič, Plestenjak, in preparation