# A Bound for the Condition of a Hyperbolic Eigenvector Matrix

Ivan Slapničar* and Krešimir Veselić†

**Abstract**

The hyperbolic eigenvector matrix is a matrix $X$ which simultaneously diagonalizes the pair $(H, J)$, where $H$ is Hermitian positive definite and $J = \mathrm{diag}(\pm 1)$ such that $X^* H X = \Delta$ and $X^* J X = J$. We prove that the spectral condition of $X$, $\kappa(X)$, is bounded by $\kappa(X) \leq \sqrt{\min \kappa(D^* H D)}$, where the minimum is taken over all non-singular matrices $D$ which commute with $J$. This bound is attainable and it can be simply computed. Similar results hold for other signature matrices $J$, like in the discretized Klein-Gordon equation.

## 1 Introduction

We are considering the hyperbolic eigenvalue problem

$$H\mathbf{x} = \lambda J \mathbf{x}, \tag{1}$$

where $H$ is a $n \times n$ Hermitian positive definite matrix, and $J = \mathrm{diag}(\pm 1)$. There always exists a matrix $X$ such that

$$X^* H X = \Delta, \qquad X^* J X = J, \tag{2}$$

where $\Delta$ is diagonal positive definite matrix. Since $H$ is positive definite, the pair $(H, J)$ is regular by definition from [9, Definition VI.1.2], so the existence of $X$ follows from [9, Theorem VI.1.15] and [9, Corollary VI.1.19]. The matrix $X$ is also called $J$-unitary. Obviously, the $i$-th eigenvalue of the problem (1) is given by

$$\lambda_i = \Delta_{ii} J_{ii},$$

and the $i$-th column of $X$ is the corresponding eigenvector. We call such eigenvectors hyperbolic, or $J$-unitary, contrary to the standard unitary eigenvectors of the problem $H\mathbf{x} = \lambda \mathbf{x}$. The matrix $X$ is also called a hyper-exchange matrix with respect to the signature matrix $J$ [5].

The matrix $X$ also appears in other linear algebra problems. For example, $X$ is the eigenvector matrix of the matrix $JH$,

$$X^{-1}(JH)X = JX^*J(JH)X = J\Delta.$$

*University of Split, Faculty of Electrical Engineering, Mechanical Engineering, and Naval Architecture, R. Boškovića b.b, 21000 Split, Croatia, e-mail: Ivan.Slapnicar@fesb.hr. Part of this work was done while the author was visiting Fernuniversität Hagen. The author was also supported by the grant 037012 from the Croatian Ministry of Science and Technology.

†Fernuniversität Hagen, Lehrgebiet Mathematische Physik, Postfach 940, D-58084 Hagen, Germany, e-mail: Kresimir.Veselic@FernUni-Hagen.de.

Also, $X$ is right singular vector matrix of the *hyperbolic singular value decomposition* (HSVD) of the pair $(G, J)$. The HSVD for the full column-rank $G$ is defined as

$$G = U\Sigma X^*,$$

where

$$U^*U = I, \quad X^*JX = J, \quad \Sigma = \mathrm{diag}(\sigma_i), \quad \sigma_i > 0.$$

Such HSVD is used in the highly accurate algorithm for the eigenvalue decomposition of a possibly indefinite symmetric (Hermitian) matrix $A$ [11, 7]: the idea is to factorize $A$ as $A = GJG^*$ [8] and then compute the HSVD of the pair $(G, J)$. Further, HSVD and its variant for the full row-rank $G$ is a suitable way to compute the eigenvalue decomposition of the difference of two outer products [15, 5], and the condition of $X$ appears in the perturbation bounds for the eigenvalues of the non-singular matrix $GJG^*$ [13]. Also, note that hyperbolic eigenvalue problems with other signature matrices (c.f. section 3) arise within some Lanczos-type algorithms for non-symmetric matrices [4].

In the paper $\|\cdot\|$ denotes the the spectral matrix norm, and $\kappa(A)$ denotes the condition of a non-singular matrix $A$,

$$\kappa(A) = \|A\| \, \|A^{-1}\|.$$

The hyperbolic eigenvector matrix has two important properties:

1. All matrices which perform the simultaneous diagonalization (2) have the same condition [11].
2. $\kappa(X) = \|X\|^2$. Moreover, the singular values of $X$ come in pairs of reciprocals, $\{\sigma, 1/\sigma\}$.

The condition $\kappa(X)$ can be expressed in terms of a Hermitian matrix which is associated to the problem (1). Let us define the spectral absolute value $|A|_S$ of the Hermitian matrix $A$ as its positive definite polar factor. That is, if $A = Q\Lambda Q^*$ is the eigenvalue decomposition of $A$, then

$$|A|_S = Q|\Lambda|Q^* = \sqrt{A^2}.$$

**Theorem 1** *Let $H = Z^*Z$ be some factorization of $H$. Then*

$$\kappa(X) = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}ZZ^*\mathbf{x}}{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}}{\mathbf{x}ZZ^*\mathbf{x}}.$$

*Proof.* The first equality was proved in [13], and the second equality follows because the eigenvalues of $XX^*$ come in the pairs of reciprocals. ∎

Note that the spectral absolute value appears naturally in the relative perturbation bounds for Hermitian and normal matrices [14, 1].

Since the maxima in Theorem 1 are not easy to compute, it is of interest to obtain a simpler bound for $\kappa(X)$. Veselić [12] recently proved that

$$\kappa(X) \leq \min_{D\in\mathcal{D}} \kappa(D^*HD),$$

where $\mathcal{D}$ is the set of all non-singular matrices which commute with $J$. In this paper we shall prove a better bound, namely

$$\kappa(X) \leq \sqrt{\min_{D\in\mathcal{D}} \kappa(D^*HD)}. \tag{3}$$

We shall also show for which matrices $D$ the minimum is attained, and for which matrices $H$ the bound itself is attained.

The rest of the paper is organized as follows: in section 2 we prove the above results, and in section 3 we apply our results to eigenvalue problems with other signature matrices, and in particular to the discretized Klein-Gordon equation and some Hamiltonian systems.

## 2 Bound for $\kappa(X)$

We shall prove the bound (3) for $\kappa(X)$ in two stages: we shall first analyze the case when the bound is an equality, and then prove the bound itself. From now on we assume without loss of generality that $J$ has the form

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix}, \tag{4}$$

which is easily achieved by permutation. Since all results of this section are trivial if in $m = 0$ or $m = n$, we assume that $0 < m < n$. Also,

$$\mathcal{D} = \{D = D_1 \oplus D_2 : D_1 \in \mathbf{C}^{m,m}, D_2 \in \mathbf{C}^{n-m,n-m}, \text{ nonsingular}\}.$$

will denote the set of all non-singular matrices which commute with $J$ from (4). To prove our results we need the following theorem which appeared in [3] (see also [2]).

**Theorem 2** *[3, Theorem 2] Let*

$$H = \begin{bmatrix} I_m & \Psi \\ \Psi^* & I_{n-m} \end{bmatrix} \tag{5}$$

*be positive definite. Then*

$$\kappa(H) = \min_{D \in \mathcal{D}} \kappa(D^* H D).$$

The following theorem shows that the bound (3) becomes an equality for matrices of the form (5).

**Theorem 3** *Let $J$ and $H$ be given by (4) and (5), respectively. Let $X$ be some matrix which diagonalizes the pair $(H, J)$ according to (2). Then*

$$\kappa(X) = \sqrt{\kappa(H)} = \sqrt{\min_{D \in \mathcal{D}} \kappa(D^* H D)}.$$

*Proof.* The second equality follows from Theorem 2. Let us construct one particular $X$. Let $U^* \Psi V = \Sigma = \operatorname{diag}(\sigma_i)$ be the singular value decomposition of $\Psi$. Set

$$W = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

and $H_1 = W^* H W$. Then $W^* J W = J$ and

$$H_1 = \begin{bmatrix} I_m & \Sigma \\ \Sigma^T & I_{n-m} \end{bmatrix}.$$

Since $H$ is positive definite we have $\sigma_i \leq \|\Psi\| < 1$, and

$$\kappa(H_1) = \kappa(H) = \frac{1 + \sigma_{\max}}{1 - \sigma_{\max}}.$$

Let $R$ be the matrix which diagonalizes the pair $(H_1, J)$ according to (2), and let $k = \min\{m, n-m\}$. Then

$$R = \begin{bmatrix} c_1 & & & & & s_1 & & & & \\ & c_2 & & & & & s_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & c_k & & & & & s_k & \\ & & & & I_{m-k} & & & & & 0 \\ s_1 & & & & & c_1 & & & & \\ & s_2 & & & & & c_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & s_k & & & & & c_k & \\ & & & & 0 & & & & & I_{n-m-k} \end{bmatrix} \tag{6}$$

Here, $c_i$ and $s_i$ are hyperbolic sines and cosines computed as follows: if $\sigma_i = 0$, then $c_i = 1$, $s_i = 0$; otherwise

$$\begin{aligned} t_i &= -\frac{\sigma_i}{1 + \sqrt{1 - \sigma_i^2}}, \\ c_i &= \frac{1}{\sqrt{1 - t_i^2}}, \\ s_i &= c_i \cdot t_i. \end{aligned}$$

Now we have $X = WR$, and, since $W$ is unitary, $\kappa(X) = \kappa(R)$. A straightforward computation shows that

$$\kappa(R) = \frac{1 + \max|t_i|}{1 - \min|t_i|} = \sqrt{\frac{1 + \sigma_{\max}}{1 - \sigma_{\max}}} = \sqrt{\kappa(H)}.$$

The theorem now follows from the fact that all $X$ which perform the required diagonalization have the same condition. ■

The aim of our main theorem is twofold: to prove the bound (3) and to define the matrix $D$ for which the minimum is attained.

**Theorem 4** *Let $J$ be given by (4) and let*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \tag{7}$$

*be partitioned accordingly. Let*

$$D = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & G_2^{-1} \end{bmatrix},$$

*where $H_{ii} = G_i^* G_i$ is some factorization of $H_{ii}$ for $i \in \{1, 2\}$, respectively. Let $\widehat{H} = D^* H D$ and let $X$ be the matrix which diagonalizes the pair $(H, J)$ according to (2). Then*

$$\kappa(X) \le \sqrt{\kappa(\widehat{H})} = \sqrt{\min_{D \in \mathcal{D}} \kappa(D^* H D)}. \tag{8}$$

*Proof.* The equality in (8) follows from Theorem 3 since $\widehat{H}$ has the form (5) with $\Psi = G_1^{-*} H_{12} G_2^{-1}$. Since all $X$ which perform the diagonalization (2) have the same condition, it remains to prove the inequality in (8) for one particular $X$ which we shall now construct.

Let $\widehat{X} = WR$ be the matrix which diagonalizes the pair $(\widehat{H}, J)$ as in the proof of Theorem 3, and let $k = \min\{m, n-m\}$. Then

$$\widehat{X}^* \widehat{H} \widehat{X} = \text{diag} = S^2 = T^2 \oplus I_{m-k} \oplus T^2 \oplus I_{n-m-k},$$

where $T^2 = \text{diag}(1 + \sigma_i \cdot t_i)$. Special forms of $R$ from (6) and $S$ imply that they commute. Set

$$Z = DWRS^{-1}. \tag{9}$$

Since $Z^* H Z = I$ we conclude that the eigenvalue decomposition of the matrix $Z^* J Z$ is given by

$$Z^* J Z = QJ\Delta^{-1}Q^*, \tag{10}$$

where $Q$ is unitary, and $\Delta$ is given by (2). Therefore, the matrix

$$X = ZQ\Delta^{1/2}$$

performs the required diagonalization of the pair $(H, J)$. Since $\kappa(X) = \|X\|^2$ we have

$$\kappa(X) = \lambda_{\max}(ZQ\Delta Q^* Z^*). \tag{11}$$

Inverting (10) gives

$$Z^{-1} J Z^{-*} = Q\Delta J Q^* = Q\Delta Q^* QJQ^*,$$

and inserting this expression for $Q\Delta Q^*$ into (11) gives

$$\kappa(X) = \lambda_{\max}(ZZ^{-1}JZ^{-*}QJQ^*Z^*) = \lambda_{\max}(QJQ^*Z^*JZ^{-*}). \tag{12}$$

In the last equality we have used the fact that for any square matrices $A$ and $B$, the matrices $AB$ and $BA$ have the same eigenvalues. ¿From (9), by using $J$-orthogonality of $R$, the fact that $R$ and $S$ commute, and the fact that $D$, $W$ and $S$ commute with $J$, we have

$$Z^* J Z^{-*} = S^{-1}R^*W^*D^*JD^{-*}W^{-*}R^{-*}S = R^*JR^{-*} = R^2 J.$$

By inserting this into (12), by using the fact that $\lambda_i(A) \leq \|A\|$ for any matrix $A$, and by using unitarity of $Q$ and $J$, Theorem 3 finally gives

$$\kappa(X) = \lambda_{\max}(QJQ^*R^2J) \leq \|QJQ^*R^2J\| = \|R^2\| = \kappa(R) = \kappa(\widehat{X}) = \sqrt{\kappa(\widehat{H})},$$

as desired. ∎

# 3 Other signature matrices

Theorem 4 yields as a corollary similar results for the eigenvalue problems with other Hermitian and skew-Hermitian signature matrices. Let us consider the simultaneous diagonalization of the pair $(H, J_S)$ where $H$ is positive definite matrix and the signature matrix is given by

$$J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{or} \quad J_S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

In both cases we seek $X$ such that $X^* J_S X = J_S$ and

$$X^* H X = \begin{bmatrix} \Delta & \Delta_1 \\ \Delta_1 & \Delta \end{bmatrix} \quad \text{or} \quad X^* H X = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}, \tag{13}$$

respectively, where $\Delta$ and $\Delta_1$ are diagonal matrices. These forms readily contain the eigenvalues of the problem

$$H\mathbf{x} = \lambda J_S \mathbf{x}, \tag{14}$$

and keep real arithmetic whenever possible. Let us set

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \qquad \text{or} \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix},$$

respectively, where $i$ is the imaginary unit. Then we have

$$U^* J_S U = J \qquad \text{or} \qquad U^* J_S U = iJ,$$

respectively, where $J = I \oplus (-I)$. Now set $\widetilde{H} = U^* H U$, and let $\widetilde{X}$ be the matrix which diagonalizes the pair $(\widetilde{H}, J)$ as in (2) such that $\widetilde{X}^* J \widetilde{X} = J$ and

$$\widetilde{X}^* \widetilde{H} \widetilde{X} = \begin{bmatrix} \Delta + \Delta_1 & 0 \\ 0 & \Delta - \Delta_1 \end{bmatrix} \qquad \text{or} \qquad \widetilde{X}^* \widetilde{H} \widetilde{X} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

respectively. Then $X = U \widetilde{X} U^*$ performs the diagonalization (13), and, by applying Theorem 4 to the above reduction, we have

$$\kappa(X) = \kappa(\widetilde{X}) \leq \sqrt{\min_{D \in \widetilde{\mathcal{D}}} \kappa(D^* H D)}, \tag{15}$$

where $\widetilde{\mathcal{D}}$ is the set of all non-singular matrices which commute with the respective $J_S$.

The reduction (13) for the first choice of $J_S$ appears in the case of the discretized Klein-Gordon equation (c.f. [10]) which consists of the quadratic eigenvalue problem

$$(\lambda^2 - 2\lambda V + V^2 - Z^2)\psi = 0,$$

where $Z$ and $V$ are real symmetric matrices, $Z$ is positive definite, and $\|VZ^{-1}\| < 1$. This eigenvalue problem is equivalent to the problem (14) with

$$H = \begin{bmatrix} L^* L & L^{-1} V L \\ L^* V L^{-*} & L^* L \end{bmatrix}, \qquad J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $Z = LL^*$ is some factorization of $Z$. By taking

$$D_0 = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}$$

which commutes with $J_S$, the relation (15) implies that the condition of the matrix $X$ which performs the required diagonalization of the pair $(H, J_S)$ is bounded by

$$\kappa(X) \leq \sqrt{\kappa(D_0^* H D_0)} = \sqrt{\frac{1 + \|VZ^{-1}\|}{1 - \|VZ^{-1}\|}}.$$

The reduction (13) for the second choice of $J_S$ comes in solution of certain Hamiltonian systems. It is also part of the highly accurate eigenvalue decomposition algorithm for skew-symmetric matrices [6]. If $H$ is partitioned according to this $J_S$ as in (7), then the minimum in (15) is attained for

$$D = \begin{bmatrix} G_1 + G_2 & -i(G_1 - G_2) \\ i(G_1 - G_2) & G_1 + G_2 \end{bmatrix},$$

where

$$G_1^* G_1 = \frac{1}{2} \times [H_{11} + H_{22} + i(H_{12} - H_{12}^*)],$$
$$G_2^* G_2 = \frac{1}{2} \times [H_{11} + H_{22} + i(H_{12}^* - H_{12})],$$

and (15) is an equality if $H_{11} + H_{22} = 2I$ and $H_{12} = H_{12}^*$.

Similar bounds can be easily derived for a matrix $X$ which diagonalizes any pair $(H, J)$, where $H$ is positive definite and $J$ satisfies merely the condition

$$J = J^* = J^{-1} \qquad \text{or} \qquad J = -J^* = -J^{-1}.$$

# References


[1] S. C. Eisenstat and I. C. F. Ipsen, Absolute perturbation bounds for matrix eigenvalues imply relative bounds, Technical report, Dept. of Computer Science, Yale University, 1997.

[2] S. C. Eisenstat, J. W. Lewis, and M. H. Schultz, Optimal block-diagonal scaling of block 2-cyclic matrices, *Linear Algebra. Appl.*, 44:181-186 (1982).

[3] L. Elsner, A note on optimal block-scaling of matrices, *Numer. Math.*, 44:127-128 (1984).

[4] R. W. Freund, Lanczos-type algorithms for structured non-Hermitian eigenvalue problems, in *Proceedings of the Cornelius Lanczos International Centenary Conference*, ( J. D. Brown, M. T. Chu, D. C. Ellison and R. J. Plemmons, Eds.), SIAM, Philadelphia, 1994.

[5] R. Onn, A. O. Steinhardt, and A. Bojanczyk, Hyperbolic singular value decomposition and applications, *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 1575-1588 (1991).

[6] E. Pietzsch, Genaue Eigenwertberechnung nichtsingulärer schiefsymmetrischer Matrizen, Ph.D. Thesis, Fernuniversität Hagen, Germany, 1993.

[7] I. Slapničar, Accurate Symmetric Eigenreduction by a Jacobi method, Ph.D. Thesis, Fernuniversität Hagen, Germany, 1992.

[8] I. Slapničar, Componentwise analysis of direct factorization of real symmetric and Hermitian matrices, *Linear Algebra Appl.*, 272:227-275 (1998).

[9] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic press, Boston, 1990.

[10] K. Veselić, A spectral theory for the Klein-Gordon equation with an external electrostatic potential, *Nucl. Phys.*, A147:215-224 (1970).

[11] K. Veselić, A Jacobi eigenreduction algorithm for definite matrix pairs, *Numer. Math.*, 64:241-269 (1993).

[12] K. Veselić, On the condition of $J$-orthonormal eigenvectors, Technical report, Fernuniversität Hagen, Germany, 1994.

[13] K. Veselić, Perturbation theory for the eigenvalues of indefinite Hermitian matrices, Technical report, Fernuniversität Hagen, Germany, 1998.

[14] K. Veselić and I. Slapničar, Floating-point perturbations of Hermitian matrices, *Linear Algebra Appl.*, 195:81–116 (1993).

[15] H. Zha, A note on the existence of the hyperbolic singular value decomposition, *Linear Algebra Appl.*, 240:199-205 (1996).